Automorphisms of affine varieties

Aleksandr Perepechko

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Présentée par

Alexander PEREPECHKO

Thèse dirigée par **Mikhail ZAIDENBERG**
et codirigée par **Ivan V. ARZHANTSEV**

préparée au sein de **l'Institut Fourier et de la Faculté de Mécanique et de Mathématiques**
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devant le jury composé de :

M. Ivan V. ARZHANTSEV
Professeur, Université d'État de Moscou, Russie, Co-Directeur de thèse
M. Jean–Philippe FURTER
Maître de conférences, Université de La Rochelle, Rapporteur
M. Stéphane LAMY
Professeur, Université Toulouse III, Examinateur
M. Laurent MANIVEL
DR CNRS, Université Grenoble I, Président
Mme. Lucy MOSER–JAUSLIN
Professeur, Université de Bourgogne, Rapporteur
M. Mikhail ZAIDENBERG
Professeur, Université Grenoble I, Directeur de thèse

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Contents

Chapitre 1

Introduction française

La lettre $\mathbb{K}$ désigne un corps algébriquement clos. Dans toutes les sections de cette thèse, à l'exception du chapitre 3 et du paragraphe 5.2, nous supposerons que $\mathbb{K}$ est de caractéristique zéro.

Cette thèse est divisée en deux parties indépendantes.

Dans la première partie, nous considérons comme objet d'étude principal la classe des monoïdes d'endomorphismes ainsi que celle des groupes d'automorphismes d'une algèbre de dimension finie.

Dans la seconde partie, nous étudions la propriété de flexibilité pour les variétés algébriques affines. Comme application, nous décrivons de nouvelles classes d'exemples de variétés algébriques avec un groupe d'automorphismes opérant infiniment transitivement dans le lieu lisse.

1.1 Partie I

1.1.1 Monoïdes algébriques affines et algèbres de dimension finie

Rappelons que tout groupe algébrique affine peut être réalisé comme un sous-groupe Zariski fermé du groupe linéaire $\mathrm{GL}(V)$ où V est un espace vectoriel de dimension finie. De façon analogue, tout monoïde affine est isomorphe à un sous-monoïde Zariski fermé du monoïde des endomorphismes d'un espace vectoriel de dimension finie (voir [17, Theorem 3.8] ou aussi [3, Lemma 1.11]).

Soit A une algèbre de dimension finie sur le corps $\mathbb{K}$. Cela veut dire que A est un espace vectoriel de dimension finie muni d'une application bilinéaire $\alpha\colon A\times A \to A$. On notera qu'en général l'application α ne possède pas les propriétés d'associativité ou de commutativité. Dans la suite, nous désignerons par $\mathrm{vect}(A)$ l'espace vectoriel sous-jacent à l'algèbre A. Le groupe des automorphismes de A est noté $\mathrm{Aut}\, A$, c'est un sous-groupe fermé de $\mathrm{GL}(\mathrm{vect}(A))$. Ainsi, $\mathrm{Aut}\, A$ est munie d'une structure naturelle de groupe algébrique. De même, le monoïde $\mathrm{End}\, A$ des endomorphismes de A est un monoïde algébrique.

Une question naturelle est de savoir si la réciproque est vraie ; c'est-à-dire, est-ce que tout groupe (resp. monoïde) algébrique affine est le groupe (resp. monoïde) des automorphismes (resp. endomorphismes) d'une algèbre de dimension finie ? N.L. Godeev et V.L. Popov ont étudié ce problème pour les groupes algébriques affines. Comme conséquence de leurs travaux on obtient le résultat suivant.

Théorème 1.1 ([8, Theorem 1]). *Soit $\mathbb{K}$ un corps algébriquement clos. Considérons G un groupe algébrique linéaire sur $\mathbb{K}$. Alors il existe une algèbre A de dimension finie sur $\mathbb{K}$ telle que le groupe algébrique $\mathrm{Aut}\, A$ est isomorphe à G.*

Dans le chapitre 3, nous proposons d'étudier le cas des monoïdes algébriques affines, voir le théorème 3.1.

1.1.2 Résolubilité des groupes d'automorphismes

Soit S une algèbre associative commutative de dimension finie sur $\mathbb{K}$. On rappelle que l'algèbre tangente du groupe algébrique $\mathrm{Aut}\, S$ est l'algèbre de Lie $\mathrm{Der}\, S$ des $\mathbb{K}$-dérivations sur S (voir [13, Ch.1, §2.3, ex. 2]). Notons $\mathrm{Aut}^\circ S$ la composante connexe de l'élément neutre (on écrira composante neutre, pour abréger) de $\mathrm{Aut}\, S$. Alors étudier la résolubilité de $\mathrm{Aut}^\circ S$ en tant que groupe est équivalent à étudier la résolubilité de $\mathrm{Der}\, S$ en tant qu'algèbre de Lie.

Une des motivations du problème de résolubilité des groupes d'automorphismes d'algèbres est la conjecture de S. Halperin énoncée ci-après.

Conjecture 1.2 (Halperin, 1987). *Soient $f_1, \ldots, f_n \in K[x_1, \ldots, x_n]$. Supposons que la $\mathbb{K}$-algèbre quotient $S = \mathbb{K}[x_1, \ldots, x_n]/(f_1, \ldots, f_n)$ soit de dimension finie et non nulle, de sorte que la sous-variété $\mathbb{V}(f_1, \ldots, f_n)$ de l'espace affine $\mathbb{K}^n$ soit d'intersection complète. Alors le groupe $\mathrm{Aut}^\circ S$ est résoluble.*

Comme réponse partielle à cette conjecture, nous avons le résultat suivant dû à H. Kraft et C. Procesi.

Théorème 1.3 (Kraft–Procesi, [11]). *Soient $f_1, \ldots, f_n \in \mathbb{K}[x_1, \ldots, x_n]$ des polynômes homogènes. Supposons que*

$$S = \mathbb{C}[x_1, \ldots, x_n]/(f_1, \ldots, f_n) \tag{1.1}$$

soit de dimension finie et non nulle. Alors le groupe $\mathrm{Aut}^\circ S$ est résoluble.

Désignons par R l'algèbre des séries formelles $\mathbb{K}[\![x_1, \ldots, x_n]\!]$ à n variables et par $\mathfrak{m}$ l'idéal maximal $(x_1, \ldots, x_n)$ de R. Soit I un idéal contenu dans $\mathfrak{m}$ tel que $S = R/I$ est une algèbre locale de dimension finie. L'idéal maximal de l'algèbre S est alors $\bar{\mathfrak{m}} = \mathfrak{m}/I$. En 2009, M. Schulze a obtenu le critère suivant.

Théorème 1.4 (Schulze, [19])**.** *Soit $S = R/I$ une algèbre locale de dimension finie, où $I \subset \mathfrak{m}^l$. Si l'inégalité*

$$\dim(I/\mathfrak{m}I) < n + l - 1 \tag{1.2}$$

est vérifiée, alors l'algèbre des dérivations $\operatorname{Der} S$ *est résoluble.*

Le prochain corollaire est une généralisation du théorème de Kraft-Procesi. En effet, les hypothèses de l'énoncé 1.3 impliquent que l'algèbre S est locale.

Corollaire 1.5 (Schulze, [19, Corollary 2])**.** *Si $S = R/(f_1, \ldots, f_n)$ est une algèbre locale provenant d'une intersection complète de $f_1, \ldots, f_n$ alors le groupe* $\operatorname{Aut}^\circ S$ *est résoluble.*

Dans la suite, nous étudions le cas des hypersurfaces ayant une singularité isolée. Soit $p \in \mathbb{K}[x_1, \ldots, x_n]$ un polynôme tel que l'origine soit une singularité isolée de l'hypersurface $H \subset \mathbb{K}^n$ d'équation $p = 0$. Dans la terminologie de G.R. Kempf [10], le *jacobien* $J(p)$ du polynôme p est le sous-espace vectoriel de $\mathbb{K}[x_1, \ldots, x_n]$ engendré par $\frac{\partial p}{\partial x_1}, \ldots, \frac{\partial p}{\partial x_n}$. L'*idéal jacobien* est l'idéal de $\mathbb{K}[x_1, \ldots, x_n]$ engendré par le jacobien de p. Le quotient $A(H) = \mathbb{K}[x_1, \ldots, x_n]/(p, J(p))$ est appelé *algèbre locale* (ou *algèbre de modules*) de l'hypersurface H, c'est une algèbre locale de dimension finie sur $\mathbb{K}$. En théorie des singularités, l'algèbre $A(H)$ est aussi appelée *l'algèbre de Tyurina.*

Il a été démontré par J. Mather et S. S.–T. Yau dans [12] que deux hypersurfaces affines complexes ayant une singularité isolée à l'origine sont localement biholomorphiquement équivalentes si et seulement si leurs algèbres de modules respectives sont isomorphes. Ainsi lorsque $\mathbb{K} = \mathbb{C}$, l'algèbre de modules permet de classifier les singularités isolées d'hypersurfaces.

Afin de déterminer les algèbres de modules parmi les algèbres locales de dimension finie, S.S.–T. Yau a introduit l'algèbre de Lie $L(H) = \operatorname{Der} A(H)$ appelée parfois *algèbre de Yau.* Il a obtenu le résultat suivant.

Théorème 1.6 (S.S.–T. Yau [24])**.** *L'algèbre de Yau $L(H)$ d'une hypersurface affine H ayant une singularité isolée à l'origine est résoluble.*

En général, deux algèbres de modules non isomorphes peuvent avoir des algèbres de Yau isomorphes. Cependant, lorsque les singularités sont simples sauf pour les types A_6 et D_5, l'algèbre de Yau détermine l'algèbre de modules (voir [7, Theorem 3.1]).

Dans [19], on montre le théorème 1.6 à partir de l'assertion 1.4 et à partir d'un résultat classique de théorie des invariants dû à G. Kempf. Le résultat de Kempf est le suivant.

Théorème 1.7 (Kempf, [10, Theorem 13])**.** *Soit G un groupe algébrique linéaire semi-simple complexe. Considérons un G-module rationnel V de dimension n sur $\mathbb{C}$. Fixons des fonctions coordonnées $x_1, \ldots, x_n$ sur V de sorte que $\mathbb{C}[V] = \mathbb{C}[x_1, \ldots, x_n]$ est l'algèbre des polynômes à n variables. Choisissons un polynôme homogène $p \in \mathbb{C}[V]$ de degré ≥ 3. Si le jacobien $J(p)$ est une partie G-stable pour l'opération naturelle de G dans $\mathbb{C}[V]$ alors il existe un polynôme invariant homogène $q \in C[V]^G$ de degré d tel que $J(p) = J(q)$.*

Dans le chapitre 4, nous introduisons la notion d'algèbre extrémale afin de déduire le théorème 1.6 du critère Schulze d'une manière directe, comme expliqué dans le paragraphe 4.3. Nous allons également présenter une preuve simplifiée du critère de Schulze et introduire un autre critère de résolubilité. Enfin, nous démontrons la conjecture de Halperin en toute généralité.

1.1.3 Résultats de la partie I

Notre premier résultat est un analogue pour les monoïdes affines algébriques du théorème 1.1. En d'autres termes, nous montrons que tout monoïde algébrique affine M peut être construit à partir du monoïde des endomorphismes d'une algèbre A de dimension finie. Cependant, il faut noter que deux différences importantes apparaissent par rapport aux travaux de Gordeev et de Popov ([8]), lorsque l'on passe du contexte des automorphismes de A à celui des endomorphismes. Tout d'abord, on ne peut pas s'attendre à ce que l'algèbre A obtenue de M soit simple, car le noyau de tout endomorphisme de A est un idéal de A. Par ailleurs, le monoïde $\mathrm{End}(A)$ a toujours un élément absorbant $\mathfrak{z} \in \mathrm{End}(A)$ qui est l'endomorphisme nul, tandis qu'en général M n'en possède pas. En résumé, nous avons le résultat suivant.

Théorème 1.8. *Soit $\mathbb{K}$ un corps algébriquement clos. Pour tout monoïde algébrique affine M défini sur $\mathbb{K}$, il existe une algèbre A de dimension finie sur $\mathbb{K}$ telle que* $\mathrm{End}(A) \cong M \sqcup \{\mathfrak{z}\}$*, où le singleton $\{\mathfrak{z}\}$ est une composante irréductible du monoïde algébrique* $\mathrm{End}(A)$.

Dans la suite, nous exposons des résultats concernant le problème de résolubilité des groupes d'automorphismes. Nous supposons que le corps $\mathbb{K}$ est algébriquement clos de caractéristique zéro. Dans la section 4.2, nous donnons une démonstration simplifiée du théorème 1.4. Le corollaire 1.10 ci-après est un critère de résolubilité pour des algèbres qui ne sont pas locales, en général. Cela nous permet d'établir la conjecture 1.2 de Halperin.

Théorème 1.9. *Soit S une algèbre de dimension finie. Considérons $\bar{\mathfrak{m}}_1, \ldots, \bar{\mathfrak{m}}_s$ ses idéaux maximaux, et fixons un entier k supérieur ou égal au maximum des longueurs des chaînes d'idéaux de S. En particulier, on peut prendre $k = \dim S$. Alors la composante neutre $\operatorname{Aut}^\circ S$ est résoluble si et seulement si pour tout $i = 1, \ldots, s$, la composante neutre $\operatorname{Aut}^\circ S_i$ du groupe des automorphismes de l'algèbre locale $S_i = S/\bar{\mathfrak{m}}_i^k$ est résoluble.*

Corollaire 1.10. *Soit $I \subset \mathbb{K}[x_1, \ldots, x_n]$ un idéal engendré par m éléments, et soit $l > 1$ un entier satisfaisant les conditions suivantes :*

- *L'algèbre quotient $S = \mathbb{K}[x_1, \ldots, x_n]/I$ est de dimension finie.*
- *Pour tout idéal maximal $\mathfrak{m} \subset \mathbb{K}[x_1, \ldots, x_n]$, soit $I \not\subset \mathfrak{m}$, soit $I \subset \mathfrak{m}^l$.*
- *On a l'inégalité $m < n + l - 1$.*

Alors la composante neutre $\operatorname{Aut}^\circ S$ est résoluble.

Dans la suite, nous introduisons et décrivons les algèbres extrémales. On notera que ces algèbres satisfont la condition limite où le critère de Schulze ne s'applique pas.

Définition 1.11. On dit qu'une algèbre locale S de dimension finie est *extrémale* si on a l'égalité $\dim I/\mathfrak{m}I = l + n - 1$.

Théorème 1.12. *Soit S une algèbre extrémale. Alors l'algèbre de Lie des dérivations $\operatorname{Der} S$ n'est pas résoluble si et seulement si S est de la forme $S = S_1 \otimes S_2$, où*

$$S_1 \cong \mathbb{K}[\![x_1, x_2]\!]/(x_1^l, x_1^{l-1}x_2, \ldots, x_1x_2^{l-1}, x_2^l) \text{ pour } l \geq 2, \tag{1.3}$$

$$S_2 \cong \mathbb{K}[\![x_3, \ldots, x_n]\!]/(w_2, \ldots, w_{n-1}), \tag{1.4}$$

et où $w_i \in \mathfrak{m}^l \cap \mathbb{K}[\![x_3, \ldots, x_n]\!]$ forment une suite régulière.

La description des algèbres extrémales avec une algèbre de Lie des dérivations non résoluble donnée dans le théorème 1.12 nous permet de déduire directement le théorème 1.6 à partir du critère de Schulze d'une manière directe, comme expliqué dans la section 4.3.

De plus, nous proposons un nouveau critère de résolubilité pour les groupes d'automorphismes. Pour cela, introduisons la notion suivante.

Définition 1.13. Pour un idéal homogène I de l'algèbre graduée R, nous désignons par I_k sa k-ième pièce graduée. On dit qu'une algèbre locale $S = R/I$ de dimension finie est *étroite* si pour tout $k \in \mathbb{Z}_{>0}$, on a l'inégalité

$$\dim I_k - \dim(\bar{\mathfrak{m}}I)_k \leq k. \tag{1.5}$$

Autrement dit, S est étroite s'il existe un sous-ensemble de générateurs homogènes de I tel que le nombre de générateurs de degré k ne dépasse pas k pour tout $k \geq 1$.

Rappelons que *l'algèbre graduée associée* à l'algèbre locale S est l'algèbre

$$\operatorname{gr} S = \mathbb{K} \oplus (\bar{\mathfrak{m}}/\bar{\mathfrak{m}}^2) \oplus (\bar{\mathfrak{m}}^2/\bar{\mathfrak{m}}^3) \oplus \dots,$$

i.e. $(\operatorname{gr} S)_i = \bar{\mathfrak{m}}^i/\bar{\mathfrak{m}}^{i+1}$. Notre critère de résolubilité est l'assertion suivante.

Théorème 1.14. *Soit S une algèbre locale de dimension finie telle que* $\operatorname{gr} S$ *est étroite. Alors l'algèbre de Lie des dérivations* $\operatorname{Der} S$ *est résoluble.*

Le critère ci-dessus est basé sur des arguments utilisés dans la démonstration du théorème 1.4. Ainsi, les théorèmes 1.4 et 1.14 donnent deux critères différents de résolubilité. Chacun d'entre eux est appliqué pour sa propre classe d'algèbres.

D'ailleurs, nous donnons une minoration de la dimension des groupes d'automorphismes.

Théorème 1.15. *Soit S une algèbre locale de dimension finie dont son idéal maximal est $\bar{\mathfrak{m}}$. Alors*

$$\dim \operatorname{Aut} S \geq \dim(\bar{\mathfrak{m}}/\bar{\mathfrak{m}}^2) \cdot \dim \operatorname{Soc} S.$$

Finalement, nous donnons un exemple d'algèbre d'Artin dont le groupe d'automorphismes est unipotent, voir 4.32.

1.2 Partie II

1.2.1 Préliminaires sur les variétés flexibles

Rappelons quelques notions fondamentales introduites dans [30] et [29].

Une opération d'un groupe G dans un ensemble A est dite *m-transitive* si l'opération induite de G dans l'ensemble des m-uplets d'éléments distincts de A est transitive. Une opération qui est m-transitive pour tout $m \in \mathbb{Z}_{>0}$ est dite *infiniment transitive*.

Soit X une variété algébrique de dimension ≥ 2 définie sur un corps algébriquement clos $\mathbb{K}$. Considérons une opération algébrique $\mathbb{G}_a \times X \to X$ du groupe additif $\mathbb{G}_a = (\mathbb{K}, +)$ du corps de base $\mathbb{K}$. Son image, disons L, de $\mathbb{G}_a$ dans le groupe des automorphismes $\operatorname{Aut} X$ est un sous-groupe unipotent à 1-paramètre. Nous désignons par $\operatorname{SAut} X$ le sous-groupe de $\operatorname{Aut} X$ engendré par tous les sous-groupes unipotents à 1-paramètre. Il est appelé le *groupe des automorphismes spéciaux*. Evidemment, $\operatorname{SAut} X$ est un sous-groupe distingué de $\operatorname{Aut} X$.

Supposons maintenant que $\mathbb{K}$ est de caractéristique zéro. Une variété algébrique affine est dite *flexible* si l'espace tangent de X en tout point lisse est engendré par les vecteurs tangents des orbites des sous-groupes unipotents à 1-paramètre. Le théorème suivant explique l'importance de la notion de flexibilité.

Théorème 1.16 ([29, Theorem 0.1]). *Soit X une variété algébrique affine de dimension ≥ 2 définie sur un corps algébriquement clos de caractéristique zéro. Alors les assertions suivantes sont équivalentes.*

1. *La variété X est flexible ;*

2. *Le groupe* $\mathrm{SAut}\, X$ *opère transitivement dans le lieu lisse X_{reg} de X ;*

3. *Le groupe* $\mathrm{SAut}\, X$ *opère infiniment transitivement dans le lieu lisse X_{reg}.*

Dans [30] on donne trois classes d'exemples de variétés flexibles : les cônes affines au-dessus des variétés de drapeaux, les variétés toriques affines non dégénérées de dimension ≥ 2, et les suspensions de variétés flexibles.

Définition 1.17. Soit Y une variété algébrique. Désignons par $\mathbb{K}[Y] = \Gamma(Y, \mathcal{O}_Y)$ l'algèbre des fonctions régulières sur Y. Une dérivation D sur $\mathbb{K}[Y]$ est dite *localement nilpotente* si pour tout $f \in \mathbb{K}[Y]$, il existe $n \in \mathbb{N}$ tel que $D^n(f) = 0$.

Pour une variété affine Y définie sur un corps de caractéristique zéro, il existe une correspondance bijective bien connue entre l'ensemble des dérivations localement nilpotentes (on écrira DLN, comme abréviation) sur $\mathbb{K}[Y]$ et l'ensemble des opérations de $\mathbb{G}_a$ dans Y. En effet, une opération algébrique $\mathbb{G}_a \times Y \to Y$ munit $\mathbb{K}[Y]$ d'une structure de $\mathbb{G}_a$–algèbre rationnelle, et le générateur infinitésimal de cette opération est une DLN D sur $\mathbb{K}[Y]$. Réciproquement, étant donnée une DLN D sur $\mathbb{K}[Y]$, le sous-groupe à 1-paramètre $\{\exp(tD) \mid t \in \mathbb{K}\}$ est un sous-groupe unipotent à 1-paramètre de $\mathrm{Aut}\, Y$, voir [43].

Si nous permettons que Y soit quasi-affine ou que le corps de base $\mathbb{K}$ soit de caractéristique quelconque, alors la correspondance entre DLN et opérations du groupe additif ne peut pas être établie en général comme dans le cas usuel ci-dessus. Dans le cas des variétés quasi-affines, un analogue du théorème 5.1 est donné dans [41, Theorem 1.11]. Tandis que pour le cas où $\mathbb{K}$ est de caractéristique quelconque, la notion de flexibilité reste à être developpée.

Pourtant, l'équivalence entre transitivité et transitivité infinie est vraie pour les variétés quasi-affines définies sur un corps algébriquement clos. La démonstration est donnée dans la section 5.2. Elle est basée sur la preuve originelle du théorème 5.1 dans [29] et sur une preuve pour les variétés quasi-affines en caractéristique zéro dans [31].

1.2.2 Préliminaires sur les torseurs universels

Les torseurs universels ont été introduits par Colliot–Thélène et Sansuc dans le contexte de la géométrie arithmétique pour étudier les points rationnels des variétés algébriques, voir [37], [38], [65]. Dans ces dernières années, ils étaient utilisés pour avoir des résultats positifs sur la conjecture de Manin portant sur la distribution des points rationnels des variétés algébriques. Cette approche a un impact important dans la géométrie torique. Pour des généralisations et des relations avec l'anneau total de coordonnées, voir [47], [33], [34], [46], [28].

Soit X une variété lisse dont les fonctions inversibles sont constantes. Supposons que le groupe des classes $\mathrm{Cl}(X)$ est un réseau de rang r, i.e., $\mathrm{Cl}(X)$ est un groupe abélien libre à r générateurs. Désignons par $\mathrm{WDiv}(X)$ le groupe des diviseurs de Weil sur X et fixons un sous-groupe $K \subset \mathrm{WDiv}(X)$ tel que l'application canonique $c\colon K \to \mathrm{Cl}(X)$ envoyant D sur sa classe $[D]$ est un isomorphisme. Nous définissons le *faisceau d'algèbres associé* à K comme étant

$$\mathcal{R} := \bigoplus_{D \in K} \mathcal{O}_X(D),$$

où la multiplication provient de la multiplication des sections homogènes dans le corps $\mathbb{K}(X)$ des fonctions rationnelles. Le faisceau $\mathcal{R}$ est un faisceau quasi-cohérent d'algèbres K-graduées normales sur $\mathcal{O}_X$. À isomorphisme près, il ne dépend pas du choix du sous-groupe $K \subset \mathrm{WDiv}(X)$, voir [28, Construction I.4.1.1]. Par ailleurs, le faisceau $\mathcal{R}$ est localement de type fini, et le spectre relatif $\widehat{X} = \mathrm{Spec}_X \mathcal{R}$ est une variété quasi-affine, voir [28, Corollary I.3.4.6].

L'anneau total de coordonnées de X est l'algèbre des sections globales

$$\mathcal{R}(X) := \bigoplus_{[D] \in \mathrm{Cl}(X)} \Gamma(X, \mathcal{O}_X(D)).$$

Nous avons $\Gamma(\widehat{X}, \mathcal{O}) \cong \mathcal{R}(X)$. L'anneau $\mathcal{R}(X)$ est factoriel et seulement les éléments constants non nuls de $\mathcal{R}(X)$ sont inversibles, voir [28, Proposition I.4.1.5]. Puisque le faisceau $\mathcal{R}$ est K-gradué, la variété $\widehat{X}$ est munie d'une opération d'un tore algébrique H de dimension $r = \mathrm{rang}\,\mathrm{Cl}(X)$. L'application $q\colon \widehat{X} \to X$ est appelée le *torseur universel* de la variété X. D'après [28, Remark I.3.2.7], le morphisme $q\colon \widehat{X} \to X$ est un fibré H-principal localement trivial. En particulier, le tore H opère librement dans $\widehat{X}$.

Supposons que l'anneau total de coordonnées $\mathcal{R}(X)$ est de type fini. Alors nous pouvons considérer *l'espace total des coordonnées* $\overline{X} := \mathrm{Spec}\,\mathcal{R}(X)$. C'est une H-variété affine factorielle. Par [28, Construction I.6.3.1], il existe une immersion ouverte H-équivariante $\widehat{X} \hookrightarrow \overline{X}$ telle que le complémentaire $\overline{X} - \widehat{X}$ est de codimension ≥ 2.

1.2.3 Résultats de la partie II

Tout d'abord, nous déduisons le théorème suivant.

Théorème 1.18. *Soient $\mathbb{K}$ un corps algébriquement clos et Y une variété algébrique quasi-affine irréductible de dimension ≥ 2 et définie sur $\mathbb{K}$. L'opération de $\operatorname{SAut} Y$ dans Y_{reg} est transitive si et seulement si elle est infiniment transitive.*

Dans le théorème 1.18 nous permettons que le corps de base $\mathbb{K}$ soit de caractéristique quelconque, mais ci-après nous supposons que $\mathbb{K}$ est algébriquement clos de caractéristique zéro. Nous construisons de nouvelles classes de variétés flexibles sur $\mathbb{K}$. Nous débutons par l'étude des cônes affines au-dessus de variétés projectives, notamment ceux des surfaces de del Pezzo. Dans le but de donner un critère de flexibilité des cônes affines, nous introduisons les définitions suivantes.

Définition 1.19 ([53, Definition 0.2])**.** On dit qu'un ouvert U d'une variété Y est un *cylindre* si $U \cong Z \times \mathbb{A}^1$, où Z est une variété lisse. Étant donné un diviseur $H \subset Y$, nous disons qu'un cylindre U est *H-polaire* si $U = Y - \operatorname{supp} D$, pour un diviseur effectif $D \in |dH|$, où $d > 0$.

Définition 1.20. On dit qu'un sous-ensemble $W \subset Y$ est *stable* par rapport au cylindre $U = Z \times \mathbb{A}^1$ si $W \cap U = \pi_1^{-1}(\pi_1(W))$, où $\pi_1 \colon U \to Z$ est la première projection du produit direct. En d'autres termes, toute $\mathbb{A}^1$-fibre du cylindre est contenue dans W ou ne rencontre pas W.

Définition 1.21. On dit qu'une variété Y est *recouverte transversalement* par des cylindres U_i, $i = 1, \ldots, r$, si $Y = \bigcup U_i$ et s'il n'existe pas de sous-ensemble non vide W stable par rapport à tous les U_i.

Dans ces termes, nous obtenons le critère suivant de flexibilité, basé sur un critère d'existence d'une opération de $\mathbb{G}_a$ donné dans [53].

Théorème 1.22. *Soient Y une variété normale projective et H un diviseur très ample sur Y. Considérons le cône affine $X = \operatorname{AffCone}_H Y \subset \mathbb{A}^{n+1}$ correspondant au plongement $Y \hookrightarrow \mathbb{P}^n$ donné par la polarisation de Y par H. S'il existe un recouvrement tranversal de Y_{reg} par des cylindres H-polaires, alors le cône affine X est flexible.*

Rappelons que toute surface de del Pezzo lisse de degré $d \in \{1, \ldots, 9\}$ à l'exception de $\mathbb{P}^1 \times \mathbb{P}^1$ peut être obtenue par éclatement de $9 - d$ points dans le plan projectif $\mathbb{P}^2$ en position générale. Nous nous intéressons principalement à leur immersions anticanoniques. Notons que les cônes affines anticanoniques au-dessus des surfaces de del Pezzo de degré ≥ 6 sont toriques, et par conséquent flexibles d'après [30]. Pour les surfaces de del Pezzo de degré 4 et 5 nous obtenons les résultats suivants.

Théorème 1.23. *Soit H un diviseur très ample sur une surface de del Pezzo Y de degré 5. Alors le cône affine $\operatorname{AffCone}_H Y$ est flexible.*

Théorème 1.24. *Soit Y une surface de del Pezzo de degré 4. Il existe un cône ouvert C dans l'espace de Neron–Severi $N^1_{\mathbb{Q}}(Y)$ tel que pour tout diviseur très ample $H \in C$, le cône affine* $\mathrm{AffCone}_H Y$ *est flexible. De plus, C contient le diviseur anticanonique $H = -K_Y$.*

En ce qui concerne les surfaces de del Pezzo de degré ≤ 3, la non-existence d'une opération du groupe additif dans les cônes affines pluri-anticanoniques a été établie, voir [36, Theorem 1.1] pour le cas de degré 3 et [54, Corollary 1.8] pour les cas de degré ≤ 2. Ainsi, cela répond aux problèmes posés dans [42] et [51].

Passons maintenant aux torseurs universaux de variétés A-recouvertes. Nous montrons que l'opération du groupe des automorphismes spéciaux dans un tel torseur est infiniment transitive.

Définition 1.25. Une variété algébrique irréductible X est dite *A-recouverte* s'il existe un recouvrement ouvert $X = U_1 \cup \ldots \cup U_r$, où chaque carte U_i est isomorphe à l'espace affine $\mathbb{A}^n$.

Un choix d'un tel recouvrement avec des isomorphismes $U_i \cong \mathbb{A}^n$ est appelé un *A-atlas* de X. Une sous-variété Z d'une variété A-recouverte est dite *linéaire*, si elle est linéaire en chaque carte, i.e. $Z \cap U_i$ est un sous-espace linéaire de $U_i \cong \mathbb{A}^n$. Toute variété A-recouverte est rationnelle, lisse et par le lemme 7.1, le groupe abélien $\mathrm{Pic}(X) = \mathrm{Cl}(X)$ est un réseau.

Donnons des exemples de variétés A-recouvertes.

(1) Toute variété torique complète lisse X est A-recouverte.

(2) Toute variété complète rationnelle lisse munie d'une opération d'un tore algébrique de complexité un est A-recouverte.

(3) Soit G un groupe algébrique linéaire semi-simple et P un sous-groupe parabolique de G. Alors la variété de drapeaux G/P est A-recouverte. En effet, un sous-groupe unipotent maximal N de G opère dans G/P avec une orbite ouverte U isomorphe à l'espace affine. Puisque G opère transitivement dans G/P, nous obtenons le recouvrement désiré.

(4) Plus généralement, toute variété sphérique lisse complète est A-recouverte, voir [35, Corollary 1.5].

(5) Il est connu que les solides de Fano $\mathbb{P}^3$, Q, V_5 et un élément de la famille V_{22} sont A-recouverts. D'après [44], il n'existe pas d'autres solides de Fano A-recouverts qui ont nombre de Picard égal à 1. En particulier, les solides de Fano V_{12}, V_{16}, V_{18} et V_4 de la classification de Iskovskikh [50] sont rationnels mais pas A-recouverts.

(6) Le produit de variétés A-recouvertes est à nouveau A-recouvert.

(7) Plus généralement, puisque tout fibré vectoriel sur $\mathbb{A}^n$ peut être trivialisé, tout espace total de fibrés vectoriels sur les variétés A-recouvertes est A-recouvert. La même chose est vraie pour leurs fibrés projectifs.

(8) Si une variété X est A-recouverte et si X' est un éclatement de X dont le centre est un point, ou plus généralement, est un sous-espace linéaire de codimension au moins 2, alors X' est à nouveau A-recouverte.

(9) En particulier, toutes les surfaces rationnelles projectives lisses sont obtenues par une suite d'eclatements de $\mathbb{P}^2$, ou de $\mathbb{P}^1 \times \mathbb{P}^1$, ou des surfaces de Hirzebruch F_n ; elles sont donc A-recouvertes.

(10) L'exemple (8) peut être généralisé comme suit. Considérons l'éclatement de X centré en un sous-espace linéaire Z. Les transformées strictes de sous-variétés linéaires de codimension au moins 2, qui ou bien contiennent Z, ou bien ne le rencontrent pas, sont encore linéaires (pour un choix approprié de A-atlas). Ainsi, nous pouvons itérer cette procédure.

Le théorème suivant donne un moyen d'associer une variété flexible à une variété A-recouverte.

Théorème 1.26. *Soit X une variété algébrique A-recouverte de dimension au moins 2, et soit $q\colon \widehat{X} \to X$ le torseur universel de X. Alors le groupe $\mathrm{SAut}(\widehat{X})$ opère dans la variété quasi-affine $\widehat{X}$ de manière infiniment transitive.*

Corollaire 1.27. *Soit X une variété algébrique A-recouverte de dimension au moins 2. Supposons que l'anneau total de coordonnées $\mathcal{R}(X)$ est de type fini. Alors l'espace total des coordonnées $\overline{X} := \operatorname{Spec} \mathcal{R}(X)$ est une variété affine factorielle, le groupe $\mathrm{SAut}(\overline{X})$ opère dans $\overline{X}$ avec une orbite ouverte O, et l'opération de $\mathrm{SAut}(\overline{X})$ dans O est infiniment transitive.*

Chapter 2

Introduction

We assume $\mathbb{K}$ to be an algebraically closed field of zero characteristic, except for Chapter 3 and Section 5.2, where an arbitrary characteristic is allowed.

In the first part of the thesis we consider endomorphism monoids and automorphism groups of finite-dimensional algebras. In the second part we study flexibility of affine varieties and describe numerous families of flexible varieties.

2.1 Part I

2.1.1 Preliminaries on monoids of endomorphisms

It is well known that every affine algebraic group is isomorphic to a Zariski closed subgroup of the general linear group $\mathrm{GL}(V)$ of a finite-dimensional vector space V. Similarly, every affine algebraic monoid is isomorphic to a Zariski closed submonoid of the monoid $\mathrm{L}(V)$ of all endomorphisms of a finite-dimensional vector space V, e.g. see [17, Theorem 3.8] or [3, Lemma 1.11].

Let A be a finite-dimensional algebra over the field $\mathbb{K}$, i.e. a finite-dimensional vector space A with a bilinear map $\alpha\colon A \times A \to A$. Note that the associativity or commutativity of the map α is not assumed. It is convenient to denote by $\mathrm{vect}(A)$ the underlying vector space of the algebra A. The automorphism group $\mathrm{Aut}\, A$ of the algebra A is a closed subgroup of the group $\mathrm{GL}(\mathrm{vect}(A))$, whence it is an affine algebraic group. Similarly, the endomorphism monoid $\mathrm{End}\, A$ is an affine algebraic monoid.

A natural question arises whether the converse is true, that is whether any affine algebraic group or monoid may be represented in this way. N.L. Gordeev and V.L. Popov considered this problem over an arbitrary field with sufficiently many elements. In the particular case of an algebraically closed field of arbitrary characteristic, the result of N.L. Gordeev and V.L. Popov is as follows.

Theorem 2.1 ([8, Theorem 1])**.** *Let $\mathbb{K}$ be an arbitrary algebraically closed field. Then for any linear algebraic group G over $\mathbb{K}$ there is a finite dimensional simple algebra A over $\mathbb{K}$ such that G is isomorphic to* $\operatorname{Aut} A$.

In Chapter 3 we consider the case of affine algebraic monoids. Our result is stated in Theorem 3.1.

2.1.2 Preliminaries on solvability of automorphism groups

Let S be a finite-dimensional (commutative and associative) algebra over an algebraically closed field $\mathbb{K}$ of characteristic zero. The automorphism group $\operatorname{Aut} S$ is an affine algebraic group with tangent algebra being the Lie algebra of derivations $\operatorname{Der} S$; see [13, Ch.1, §2.3, ex. 2]. Thus, the solvability of the identity component $\operatorname{Aut}^\circ S$ is equivalent to the solvability of the Lie algebra $\operatorname{Der} S$.

Let us start with the following conjecture proposed by S. Halperin.

Consider a finite collection of polynomials $f_1, \ldots, f_n \in \mathbb{K}[x_1, \ldots, x_n]$. Assume that the quotient $S = \mathbb{K}[x_1, \ldots, x_n]/(f_1, \ldots, f_n)$ is non-trivial and finite-dimensional. In this case S is a complete intersection and $f_1, \ldots, f_n \in \mathbb{K}[x_1, \ldots, x_n]$ is a regular sequence.

Conjecture 2.2 (Halperin, 1987)**.** *Suppose that a finite-dimensional algebra $S = \mathbb{C}[x_1, \ldots, x_n]/(f_1, \ldots, f_n)$ is a complete intersection. Then the identity component* $\operatorname{Aut}^\circ S$ *of the automorphism group of S is solvable.*

In the case of homogeneous polynomials, this conjecture was proved by H. Kraft and C. Procesi.

Theorem 2.3 (Kraft–Procesi, [11])**.** *Let $f_1, \ldots, f_n \in \mathbb{K}[x_1, \ldots, x_n]$ be homogeneous polynomials such that the algebra*

$$S = \mathbb{C}[x_1, \ldots, x_n]/(f_1, \ldots, f_n) \tag{2.1}$$

is finite-dimensional. Then the identity component $\operatorname{Aut}^\circ S$ *is solvable.*

We denote by R the algebra of formal power series $\mathbb{K}[\![x_1, \ldots, x_n]\!]$ and by $\mathfrak{m}$ the maximal ideal $(x_1, \ldots, x_n) \lhd R$. Let $I \subset \mathfrak{m}$ be such that $S = R/I$ is a finite-dimensional (or Artin) local algebra with the maximal ideal $\bar{\mathfrak{m}} = \mathfrak{m}/I$. In 2009, M. Schulze obtained the following criterion, which has several applications discussed below.

Theorem 2.4 (Schulze, [19])**.** *Let $S = R/I$ be a finite-dimensional local algebra, where $I \subset \mathfrak{m}^l$. If the inequality*

$$\dim(I/\mathfrak{m}I) < n + l - 1 \tag{2.2}$$

holds, then the algebra of derivations $\operatorname{Der} S$ *is solvable.*

Note that the algebra S as in Theorem 2.3 is local. Thus, the following corollary of Theorem 2.4 is a generalization of Theorem 2.3.

Corollary 2.5 (Schulze, [19, Corollary 2]). *Given a local complete intersection $S = R/(f_1, \ldots, f_n)$, the group* $\operatorname{Aut}^\circ S$ *is solvable.*

Let us now turn to the case of isolated hypersurface singularities (or IHS, for short). Let $p \in \mathbb{K}[x_1, \ldots, x_n]$ be such that the hypersurface $\{p = 0\} \subset \mathbb{K}^n$ has an isolated singularity $H = (\{p = 0\}, 0)$ at the origin. In the terminology of G.R. Kempf [10] the subspace $J(p) = \operatorname{Span}_{\mathbb{K}}\left(\frac{\partial p}{\partial x_1}, \ldots, \frac{\partial p}{\partial x_n}\right) \subset \mathbb{K}[x_1, \ldots, x_n]$ is called the *Jacobian* of p. It generates an ideal $(J(p))$ called the *Jacobian ideal* of p. The quotient $A(H) = \mathbb{K}[\![x_1, \ldots, x_n]\!]/(p, J(p))$ is called the *local algebra* or the *moduli algebra* of the IHS H. In singularity theory, $A(H)$ is also known under the name of *Tyurina algebra*. This algebra is local and finite-dimensional.

It has been proven by J. Mather and S. S.–T. Yau in [12] that two IHS are locally biholomorphically equivalent if and only if their moduli algebras are isomorphic. Thus, the finite-dimensional local algebra $A(H)$ defines the IHS H up to analytic isomorphism. In order to determine as to when a finite-dimensional local algebra is a moduli algebra of some IHS, S.S.–T. Yau [23] introduced a Lie algebra of derivations $L(H) = \operatorname{Der} A(H)$ called sometimes a *Yau algebra*. He obtained the following result.

Theorem 2.6 (S.S.–T. Yau [24]). *The algebra $L(H)$ of an IHS H is solvable.*

Note that generally the Yau algebra does not uniquely determine its moduli algebra. But for *simple* singularities this property holds with the only exception, the pair A_6 and D_5; see [7, Theorem 3.1].

In [19] M. Schulze deduces Theorem 2.6 from his criterion. He uses the following result of G. Kempf, asking whether one could avoid using this result.

Theorem 2.7 (Kempf, [10, Theorem 13]). *Let $p \in \mathbb{C}[x_1, \ldots, x_n]$ be a homogeneous polynomial of degree $d \geq 3$. Assume that the space $\mathbb{C}^n$ is endowed with a linear action of a semisimple group G. If the Jacobian $J(p) \subset \mathbb{C}[x_1, \ldots, x_n]$ is G-invariant then there exists a homogeneous G-invariant polynomial q of degree d such that $J(p) = J(q)$.*

In Chapter 4 we introduce the concept of extremal algebras. In these terms we deduce Theorem 2.6 from Schulze's criterion directly, i.e. without applying Kempf's Theorem 2.7, as explained in Section 4.3. We also suggest a simplified proof of Schulze's criterion and propose yet another solvability criterion. We finish Chapter 4 with a proof of Conjecture 2.2. So, we generalize Schulze's result to not necessarily local complete intersections.

2.1.3 Results of Part I

Our first result is an analogue of Theorem 2.1 by Gordeev and Popov for affine algebraic monoids, i.e. the realization of an arbitrary affine algebraic monoid M as the endomorphisms' monoid of some finite-dimensional algebra A. In our case two differences occur with respect to the setting of Theorem 2.1. First, we cannot expect the constructed algebra A to be simple, since the kernel of any endomorphism is an ideal of A. Second, the monoid $\mathrm{End}(A)$ always contains a zero $\mathfrak{z} \in \mathrm{End}(A)$, where $\mathfrak{z}(a) = 0$ for any $a \in A$, while M may not. Under these circumstances we obtain the following result.

Theorem 2.8. *Let $\mathbb{K}$ an algebraically closed field. For any affine algebraic monoid M over $\mathbb{K}$ there exists a finite-dimensional algebra A over $\mathbb{K}$ such that* $\mathrm{End}(A) \cong M \sqcup \{\mathfrak{z}\}$*, where a singleton $\{\mathfrak{z}\}$ is an irreducible component of the algebraic monoid* $\mathrm{End}(A)$.

Let us return to the problem of solvability of the automorphism groups. The base field $\mathbb{K}$ is assumed to be algebraically closed of characteristic zero. In Section 4.2 we provide a simplified proof of Schulze's Theorem 2.4. In Corollary 1.10 we obtain a more general solvability criterion for not necessarily local algebras. This allows us to establish Halperin's Conjecture 2.2 in full generality.

Theorem 2.9. *Let S be a finite-dimensional algebra with maximal ideals $\bar{\mathfrak{m}}_1, \ldots, \bar{\mathfrak{m}}_s$ and k be an integer at least equal to the maximal length of descending chains of ideals in S. In particular, one may take $k = \dim S$. Then the identity component* $\mathrm{Aut}^\circ S$ *is solvable if and only if the identity component* $\mathrm{Aut}^\circ S_i$ *of the local algebra $S_i = S/\bar{\mathfrak{m}}_i^k$ is solvable for any $i = 1, \ldots, s$.*

Corollary 2.10. *Let an ideal $I \subset \mathbb{K}[x_1, \ldots, x_n]$ with m generators and an integer $l > 1$ satisfy the following conditions:*

- *The quotient algebra $S = \mathbb{K}[x_1, \ldots, x_n]/I$ is finite-dimensional,*
- *For any maximal ideal $\mathfrak{m} \subset \mathbb{K}[x_1, \ldots, x_n]$, either $I \not\subset \mathfrak{m}$ or $I \subset \mathfrak{m}^l$,*
- *The inequality $m < n + l - 1$ holds.*

Then the identity component $\mathrm{Aut}^\circ S$ *is solvable.*

Further, we introduce and describe *extremal* algebras. These algebras are on the border line with respect to validity of Schulze's criterion.

Definition 2.11. We say that the finite-dimensional local algebra S is an *extremal algebra* if the equality $\dim I/\mathfrak{m}I = l + n - 1$ holds.

Theorem 2.12. *Let S be an extremal algebra. Then the algebra of derivations* $\operatorname{Der} S$ *is non-solvable if and only if S is of the form $S = S_1 \otimes S_2$, where*

$$S_1 \cong \mathbb{K}[\![x_1, x_2]\!]/(x_1^l, x_1^{l-1}x_2, \ldots, x_1x_2^{l-1}, x_2^l) \text{ for some } l \geq 2, \tag{2.3}$$
$$S_2 \cong \mathbb{K}[\![x_3, \ldots, x_n]\!]/(w_2, \ldots, w_{n-1}), \tag{2.4}$$

and where $w_2, \ldots, w_{n-1} \in \mathfrak{m}^l \cap \mathbb{K}[\![x_3, \ldots, x_n]\!]$ form a regular sequence.

The description of the extremal algebras with a non-solvable algebra of derivations given in Theorem 2.12 allows us to deduce directly Theorem 2.6 from Schulze's criterion, as explained in Section 4.3.

In addition, we suggest a new solvability criterion for the automorphism groups. To this end, let us introduce the following notion.

Definition 2.13. For a graded ideal I let us denote by I_k its kth graded component. We say that a graded local finite-dimensional algebra $S = R/I$ is *narrow* if the inequality

$$\dim I_k - \dim(\bar{\mathfrak{m}}I)_k \leq k \tag{2.5}$$

holds for all $k = 1, 2 \ldots$. In other words, S is narrow if there exists a set of homogeneous generators of I such that the number of generators of degree k does not exceed k for every $k \geq 1$.

Recall that the *associated graded algebra* of the local algebra S is the algebra

$$\operatorname{gr} S = \mathbb{K} \oplus (\bar{\mathfrak{m}}/\bar{\mathfrak{m}}^2) \oplus (\bar{\mathfrak{m}}^2/\bar{\mathfrak{m}}^3) \oplus \ldots,$$

i.e. $(\operatorname{gr} S)_i = \bar{\mathfrak{m}}^i/\bar{\mathfrak{m}}^{i+1}$. Our solvability criterion is as follows.

Theorem 2.14. *Suppose that the associated graded algebra* $\operatorname{gr} S$ *of a local finite-dimensional algebra S is narrow. Then the algebra of derivations* $\operatorname{Der} S$ *is solvable.*

This criterion is based upon a technique exploited in the proof Theorem 2.4. Thus, Theorem 2.4 and Theorem 2.14 give two different solvability criteria. Each of them is applied to its own class of algebras.

Besides, we give the following lower bound for the dimension of the automorphism group.

Theorem 2.15. *Let S be a local finite-dimensional algebra with the maximal ideal $\bar{\mathfrak{m}}$. Then*

$$\dim \operatorname{Aut} S \geq \dim(\bar{\mathfrak{m}}/\bar{\mathfrak{m}}^2) \cdot \dim \operatorname{Soc} S.$$

Finally, we provide an example of an Artin algebra with a unipotent automorphism group, see 4.32.

2.2 Part II

2.2.1 Preliminaries on flexible varieties

Recall some useful notions introduced in [29] and [30].

An action of a group G on a set A is said to be *m-transitive* if for every two m-tuples of pairwise distinct points $(a_1, \ldots, a_m)$ and $(a'_1, \ldots, a'_m)$ in A there exists $g \in G$ such that $g \cdot a_i = a'_i$ for $i = 1, \ldots, m$. An action which is m-transitive for all $m \in \mathbb{Z}_{>0}$ is called *infinitely transitive*.

Let X be an algebraic variety of dimension ≥ 2 over an algebraically closed field $\mathbb{K}$. Consider a regular action $\mathbb{G}_a \times X \to X$ of the additive group $\mathbb{G}_a = (\mathbb{K}, +)$ of the ground field $\mathbb{K}$. The image, say, L of $\mathbb{G}_a$ in the automorphism group $\operatorname{Aut} X$ is a one-parameter unipotent subgroup. We let $\operatorname{SAut} X$ denote the subgroup of $\operatorname{Aut} X$ generated by all its one-parameter unipotent subgroups. It is called a *special automorphism group*. Evidently, $\operatorname{SAut} X$ is a normal subgroup of $\operatorname{Aut} X$.

Now assume that $\mathbb{K}$ is of characteristic zero. An affine algebraic variety X is called *flexible* if the tangent space of X at any smooth point is spanned by the tangent vectors to the orbits of one-parameter unipotent group actions. The following theorem explains the significance of the flexibility concept.

Theorem 2.16 ([29, Theorem 0.1])**.** *Let X be an affine algebraic variety of dimension ≥ 2 over an algebraically closed field of characteristic zero. Then the following conditions are equivalent.*

1. *The variety X is flexible;*
2. *the group $\operatorname{SAut} X$ acts transitively on the smooth locus X_{reg} of X;*
3. *the group $\operatorname{SAut} X$ acts infinitely transitively on X_{reg}.*

The following three classes of examples of flexible varieties are described in [30]: affine cones over flag varieties, non-degenerate toric varieties of dimension ≥ 2, and suspensions over flexible varieties. .

Definition 2.17. Let Y be algebraic variety, and let $\mathbb{K}[Y] = \Gamma(Y, \mathcal{O}_Y)$ denote the algebra of regular functions on Y. A derivation D on $\mathbb{K}[Y]$ is called *locally nilpotent*, if for any $f \in \mathbb{K}[Y]$ there exists $n \in \mathbb{N}$ such that $D^n(f) = 0$.

For an affine variety Y over a field of characteristic zero there exists a canonical one-to-one correspondence between locally nilpotent derivations (or LHD's, for short) on $\mathbb{K}[Y]$ and $\mathbb{G}_a$-actions on Y. Indeed, a regular action $\mathbb{G}_a \times Y \to Y$ defines a structure of a rational $\mathbb{G}_a$-algebra on $\mathbb{K}[Y]$, and the infinitesimal generator of this action is a locally nilpotent derivation D on $\mathbb{K}[Y]$. Conversely, given a locally nilpotent derivation D on $\mathbb{K}[Y]$, the one-parameter group $\{\exp(tD) \mid t \in \mathbb{K}\}$ is a one-parameter unipotent algebraic subgroup of $\operatorname{Aut} Y$, see [43].

If we allow Y to be quasiaffine or the ground field to be of positive characteristic, then the one-to-one correspondence between LND's and $\mathbb{G}_a$-actions fails to exist. In the case of quasi-affine varieties the analogue of Theorem 5.1 is obtained in [41, Theorem 1.11]. In the case of a field $\mathbb{K}$ of positive characteristic, the notion of flexibility is yet to be developed.

Nevertheless, the equivalence of transitivity and infinite transitivity does hold for quasiaffine varieties over an algebraically closed field of arbitrary characteristic. The proof is provided in Section 5.2. It is based on the original proof of Theorem 5.1 in [29] and on a proof for quasiaffine varieties in charactestic zero in [31].

2.2.2 Preliminaries on universal torsors

Universal torsors were introduced by Colliot-Thélène and Sansuc in the framework of arithmetic geometry to investigate rational points on algebraic varieties, see [37], [38], [65]. In the last years they were used to obtain positive results on Manin's Conjecture on the distribution of rational points in algebraic varieties. Another source of interest is Cox's paper [39], where an explicit description of the universal torsor over a toric variety is given. This approach had an essential impact in toric geometry. For generalizations and relations to Cox rings, see [47], [33], [34], [46], [28].

Let X be a smooth variety with only constant invertible functions. Assume that the divisor class group $\mathrm{Cl}(X)$ is a lattice of rank r, i.e. a free abelian group with r generators. Denote by $\mathrm{WDiv}(X)$ the group of Weil divisors on X and fix a subgroup $K \subset \mathrm{WDiv}(X)$ such that the canonical map $c\colon K \to \mathrm{Cl}(X)$ sending $D \in K$ to its class $[D] \in \mathrm{Cl}(X)$ is an isomorphism. We define the *Cox sheaf* associated to K to be

$$\mathcal{R} := \bigoplus_{D \in K} \mathcal{O}_X(D),$$

where the multiplication is done via multiplying homogeneous sections in the field of rational functions $\mathbb{K}(X)$. The sheaf $\mathcal{R}$ is a quasicoherent sheaf of normal integral K-graded $\mathcal{O}_X$-algebras. Up to an isomorphism, it does not depend on the choice of the subgroup $K \subset \mathrm{WDiv}(X)$, see [28, Construction I.4.1.1]. Moreover, the sheaf $\mathcal{R}$ is locally of finite type, and the relative spectrum $\widehat{X} = \operatorname{Spec}_X \mathcal{R}$ is a quasiaffine variety, see [28, Corollary I.3.4.6].

The *Cox ring* of X is the algebra of global sections

$$\mathcal{R}(X) := \bigoplus_{[D] \in \mathrm{Cl}(X)} \Gamma(X, \mathcal{O}_X(D)).$$

We have $\Gamma(\widehat{X}, \mathcal{O}) \cong \mathcal{R}(X)$, and the ring $\mathcal{R}(X)$ is a unique factorization domain with only constant invertible elements, see [28, Proposition I.4.1.5]. Since the sheaf $\mathcal{R}$ is K-graded, the variety $\widehat{X}$ carries a natural action of an algebraic torus H of rank $r = \operatorname{rank}\operatorname{Cl}(X)$. The projection $q\colon \widehat{X} \to X$ is called the *universal torsor* over the variety X. By [28, Remark I.3.2.7], the morphism $q\colon \widehat{X} \to X$ is a locally trivial H-principal bundle. In particular, the torus H acts on $\widehat{X}$ freely.

Assume that the Cox ring $\mathcal{R}(X)$ is finitely generated. Then we may consider the *total coordinate space* $\overline{X} := \operatorname{Spec}\mathcal{R}(X)$. This is a factorial affine H-variety. By [28, Construction I.6.3.1], there is a natural open H-equivariant embedding $\widehat{X} \hookrightarrow \overline{X}$ such that the complement $\overline{X} \setminus \widehat{X}$ is of codimension at least two.

2.2.3 Results of Part II

First, we obtain the following result, see also Theorem 5.7.

Theorem 2.18. *Let $\mathbb{K}$ be an arbitrary algebraically closed field and let Y be an irreducible quasiaffine algebraic variety of dimension ≥ 2 defined over $\mathbb{K}$. Then the action of $\operatorname{SAut}Y$ on Y_{reg} is transitive if and only if it is infinitely transitive.*

In Theorem 2.18 we allow the ground field $\mathbb{K}$ to be of arbitrary characterisic, but hereafter we assume $\mathbb{K}$ to be an algebraically closed field of characteristic zero. We construct new classes of flexible varieties over $\mathbb{K}$. We start with affine cones over projective varieties, especially over del Pezzo surfaces. In order to provide a flexibility criterion for affine cones, we use the following definitions.

Definition 2.19 ([53, Definition 0.2])**.** We say that an open subset U of a variety Y is a *cylinder* if $U \cong Z \times \mathbb{A}^1$, where Z is a smooth variety. Given a divisor $H \subset Y$, we say that a cylinder U is *H-polar* if $U = Y \setminus \operatorname{supp} D$ for some effective divisor $D \in |dH|$, where $d > 0$.

Definition 2.20. We call a subset $W \subset Y$ *invariant* with respect to a cylinder $U = Z \times \mathbb{A}^1$ if $W \cap U = \pi_1^{-1}(\pi_1(W))$, where $\pi_1 : U \to Z$ is the first projection of the direct product. In other words, every $\mathbb{A}^1$-fiber of the cylinder is either contained in W or does not meet W.

Definition 2.21. We say that a variety Y is *transversally covered* by cylinders U_i, $i = 1, \ldots, s$, if $Y = \bigcup U_i$ and there is no proper subset $W \subset Y$ invariant with respect to all the U_i.

In these terms, we obtain the following criterion of flexibility, based on a criterion of existence of a $\mathbb{G}_a$-action from [53].

Theorem 2.22. *Let Y be a normal projective variety and H be a very ample divisor on Y. Consider the affine cone $X = \mathrm{AffCone}_H Y \subset \mathbb{A}^{n+1}$ corresponding to the embedding $Y \hookrightarrow \mathbb{P}^n$ provided by the polarization of Y by H. If there exists a transversal covering of Y_{reg} by H-polar cylinders, then the affine cone X is flexible.*

Recall that any smooth del Pezzo surface of degree $d \in \{1, \ldots, 9\}$ except $\mathbb{P}^1 \times \mathbb{P}^1$ can be obtained by blowing up the projective plane $\mathbb{P}^2$ in $9 - d$ points in general position. We are primarily interested in their anticanonical embeddings. Note that the anticanonical affine cones over del Pezzo surfaces of degree ≥ 6 are toric, thereby they are flexible by [30]. For del Pezzo surfaces of degree 4 and 5 we obtain the following results.

Theorem 2.23. *Let H be an arbitrary very ample divisor on a del Pezzo surface Y of degree 5. Then the corresponding affine cone $\mathrm{AffCone}_H Y$ is flexible.*

Theorem 2.24. *Let Y be a del Pezzo surface of degree 4. There is an open cone C in the Neron–Severi space $N^1_{\mathbb{Q}}(Y)$ such that for any very ample divisor $H \in C$ the affine cone $\mathrm{AffCone}_H Y$ is flexible. Moreover, C contains the anticanonical divisor $H = -K_Y$.*

As for del Pezzo surfaces of degree ≤ 3, it is proven the non-existence of $\mathbb{G}_a$-actions on the plurianticanonical affine cones, see [36, Theorem 1.1] for the case of degree 3 and [54, Corollary 1.8] for the case of degree ≤ 2. This answers the corresponding problems posed in [42] and [51].

Let us turn to the universal torsors over A-covered varieties. We prove that the action of the special automorphism group on such a torsor is infinitely transitive.

Definition 2.25. An irreducible algebraic variety X is said to be *A-covered* if there is an open covering $X = U_1 \cup \ldots \cup U_r$, where every chart U_i is isomorphic to an affine space $\mathbb{A}^n$.

A choice of such a covering together with isomorphisms $U_i \cong \mathbb{A}^n$ is called an *A-atlas* of X. A subvariety Z of an A-covered variety X is called *linear* with respect to an A-atlas, if it is linear in all charts, i.e. $Z \cap U_i$ is a linear subspace in $U_i \cong \mathbb{A}^n$. Any A-covered variety is rational, smooth, and by Lemma 7.1 the group $\mathrm{Pic}(X) = \mathrm{Cl}(X)$ is finitely generated and free.

Let us give examples of A-covered varieties.

(1) Every smooth complete toric variety X is A-covered.

(2) Every smooth rational complete variety with a torus action of complexity one is A-covered.

(3) Let G be a semisimple algebraic group and let P be a parabolic subgroup of G. Then the flag variety G/P is A-covered. Indeed, a maximal unipotent subgroup N of G acts on G/P with an open orbit U isomorphic to an affine space. Since G acts on G/P transitively, we obtain the desired covering.

(4) More generally, every complete smooth spherical variety is A-covered, see [35, Corollary 1.5].

(5) The Fano threefolds $\mathbb{P}^3$, Q, V_5 and an element of the family V_{22} are known to be A-covered. Moreover, there are no other types of A-covered Fano threefolds of Picard number 1 by [44]. In particular, the Fano threefolds V_{12}, V_{16}, V_{18} and V_4 from Iskovskikh's classification [50] are rational but not A-covered.

(6) The product of two A-covered varieties is again A-covered.

(7) More generally, since every vector bundle over $\mathbb{A}^n$ can be trivialized, the total spaces of vector bundles over A-covered varieties are A-covered. The same holds for their projectivizations.

(8) If a variety X is A-covered and X' is a blow up of X with center at a point or, more generally, at a linear subvariety of codimension at least 2, then X' is again A-covered.

(9) In particular, all smooth projective rational surfaces are obtained either from $\mathbb{P}^2$, $\mathbb{P}^1 \times \mathbb{P}^1$ or from the Hirzebruch surfaces F_n by a sequence of blow ups, and so they are A-covered.

(10) Example (8) can be generalized as follows. Let a linear subvariety Z be the center of a blow up. The strict transforms of linear subvarieties of codimension at least 2, which either contain Z or do not meet it, are linear again (with the choice of an appropriate A-atlas). Hence, we may iterate this procedure.

The following theorem provides a way to associate a flexible quasi-affine variety to any A-covered variety.

Theorem 2.26. *Let X be an A-covered algebraic variety of dimension at least 2 and $q\colon \widehat{X} \to X$ be the universal torsor over X. Then the group $\mathrm{SAut}(\widehat{X})$ acts on the quasiaffine variety $\widehat{X}$ infinitely transitively.*

Corollary 2.27. *Let X be an A-covered algebraic variety of dimension at least 2. Assume that the Cox ring $\mathcal{R}(X)$ is finitely generated. Then the total coordinate space $\overline{X} := \mathrm{Spec}\, \mathcal{R}(X)$ is a factorial affine variety, the group $\mathrm{SAut}(\overline{X})$ acts on $\overline{X}$ with an open orbit O, and the action of $\mathrm{SAut}(\overline{X})$ on O is infinitely transitive.*

Part I

Transformations of finite-dimensional algebras

Chapter 3

Monoids of endomorphisms

3.1 Introduction

In this chapter we assume $\mathbb{K}$ to be an algebraically closed field of arbitrary characteristic. Recall that an *affine algebraic semigroup* is an affine variety M over $\mathbb{K}$ with an associative product $\mu\colon M \times M \to M$, which is a morphism of algebraic varieties. Denote an element $\mu(a, b)$ by ab. A semigroup is called a *monoid* if it contains an identity element $e \in M$ such that $em = me = m$ for any $m \in M$. An element $0 \in M$ is called *zero* if $0m = m0 = 0$ for any $m \in M$. Obviously, a monoid cannot contain more than one zero. It is well known that every affine algebraic monoid is isomorphic to a Zariski closed submonoid of the monoid $\mathrm{L}(V)$ of all linear operators on some finite-dimensional vector space V, e.g. see [17, Theorem 3.8] or [3, Lemma 1.11]. A systematic account of the theory of affine algebraic monoids is given in [16] and [17]. A classification of irreducible affine monoids, whose unit group is reductive, is obtained in [18] and [20].

Let A be a finite-dimensional algebra over the field $\mathbb{K}$, i.e. a finite-dimensional vector space A with a bilinear map $\alpha\colon A \times A \to A$. Note that the associativity or commutativity of the map α is not assumed. It is convenient to denote by $\mathrm{vect}(A)$ the underlying vector space of an algebra A. By an ideal of an algebra A we mean a two-sided ideal. An algebra A is called *simple* if it does not contain proper ideals. The set of all endomorphisms of A,

$$\mathrm{End}(A) := \{\phi \in \mathrm{L}(\mathrm{vect}(A)) \mid \alpha(\phi(a), \phi(b)) = \phi(\alpha(a, b)) \text{ for } a, b \in A\},$$

is a monoid with respect to composition. It is easy to check that this monoid is Zariski closed in $\mathrm{L}(\mathrm{vect}(A))$, therefore it is an affine algebraic monoid.

It is shown in [8] that any affine algebraic group can be realized as the group of automorphisms of some finite-dimensional simple algebra. In this chapter we propose a similar realization of an arbitrary affine algebraic monoid M as the endomorphisms' monoid of a finite-dimensional algebra A. In this case two differences occur. First, we cannot assume that A is simple, since the kernel of any endomorphism is an ideal of A. Second, the monoid $\mathrm{End}(A)$ always contains a zero $\mathfrak{z} \in \mathrm{End}(A)$, $\mathfrak{z}(a) = 0$ for any $a \in A$, while M may not. Under these circumstances we obtain the following result.

Theorem 3.1. *For any affine algebraic monoid M over $\mathbb{K}$ there exists a finite-dimensional algebra A over $\mathbb{K}$ such that $\mathrm{End}(A) \cong M \sqcup \{\mathfrak{z}\}$, where a singleton $\{\mathfrak{z}\}$ is an irreducible component of the algebraic monoid $\mathrm{End}(A)$.*

In particular, if M is an affine algebraic group, then there exists an algebra A such that $\mathrm{Aut}(A) \cong M$ (see [8]).

Example 3.2. Let us consider the monoid $M = \mathrm{L}(V)$ for a finite-dimensional space V. Then we may take the algebra A constructed in the following way. First, let e be a left identity of A and

$$\mathrm{vect}(A) := \langle e \rangle \oplus V,$$

where $\langle X \rangle$ stands for the linear span of a set X. Next, for any $v, w \in V$ define their product by $v \cdot w = 0$, $v \cdot e = \lambda v$, where λ is a fixed constant in $\mathbb{K} \setminus \{0, 1\}$. Taking into account the equalities $e \cdot v = v$ and $e \cdot e = e$, we obtain the multiplication table of A.

Note that any endomorphism sends e to e or 0, since these two are the only idempotents of A. In this way, the reader will easily prove that $\mathrm{End}(A) \cong \mathrm{L}(V) \sqcup \{\mathfrak{z}\}$, where $\mathfrak{z}$ is as in Theorem 3.1.

Example 3.3. Assume $\mathrm{char}\,\mathbb{K} \neq 2$. Consider a two-dimensional space V over $\mathbb{K}$ with a basis $\{v_1, v_2\}$ and the exterior algebra $\Lambda(V)$ with a basis $\{1, v_1, v_2, v_1 \wedge v_2\}$. Let us take a monoid $M \subset \mathrm{L}(\mathrm{vect}(\Lambda(V)))$,

$$M := \left\{ \left(\begin{array}{cccc} 1 & 0 & 0 & 0 \\ 0 & b_{11} & b_{12} & 0 \\ 0 & b_{21} & b_{22} & 0 \\ 0 & c_1 & c_2 & d \end{array} \right) \middle| \, d = \det \left(\begin{array}{cc} b_{11} & b_{12} \\ b_{21} & b_{22} \end{array} \right), b_{ij}, c_i \in \mathbb{K} \right\}.$$

One can show that M acts on $\Lambda(V)$ by endomorphisms. Moreover, $\mathrm{End}(\Lambda(V)) = M \sqcup \{\mathfrak{z}\}$. More generally, a similar equation holds for the exterior algebra of an arbitrary space.

The proof of Theorem 3.1 is done of two steps. First, for every finite-dimensional space U and its subspace S we construct a finite-dimensional algebra A such that $\mathrm{End}(A)$ is isomorphic to $\mathrm{L}(U)_S \sqcup \{\mathfrak{z}\}$, where $\mathrm{L}(U)_S$ is the normalizer of some vector subspace S of a special $\mathrm{L}(U)$-module. Second, an arbitrary affine algebraic monoid M is represented as $\mathrm{L}(U)_S$ for appropriate U and S.

3.2 Some special algebras

In this section we define and study some finite-dimensional algebras that will be useful later on.

3.2.1 Algebra $A(V, S)$

Let V be a nonzero finite-dimensional vector space. Denote by $\mathrm{T}(V)$ the tensor algebra of V and by $\mathrm{T}(V)_+$ its maximal homogeneous ideal

$$\mathrm{T}(V)_+ := \bigoplus_{i\geq 1} V^{\otimes i}, \tag{3.1}$$

endowed with the natural $\mathrm{L}(V)$-structure,

$$g \cdot t_i := g^{\otimes i}(t_i), \quad g \in \mathrm{L}(V),\ t_i \in V^{\otimes i}. \tag{3.2}$$

Thus, $\mathrm{L}(V)$ acts on $\mathrm{T}(V)_+$ faithfully by endomorphisms. Therefore we may identify $\mathrm{L}(V)$ with the corresponding submonoid of $\mathrm{End}(\mathrm{T}(V)_+)$.

Fix an integer $r > 1$. For an arbitrary subspace $S \subset V^{\otimes r}$ we define

$$I(S) := S \oplus (\bigoplus_{i>r} V^{\otimes i}). \tag{3.3}$$

It is an ideal of $\mathrm{T}(V)_+$. Define $A(V, S)$ as the quotient algebra modulo this ideal,

$$A(V, S) := \mathrm{T}(V)_+/I(S). \tag{3.4}$$

Then

$$\mathrm{vect}(A(V, S)) = (\bigoplus_{i=1}^{r-1} V^{\otimes i}) \oplus (V^{\otimes r}/S). \tag{3.5}$$

We may consider $\mathrm{L}(V)_S := \{\phi \in \mathrm{L}(V) \mid \phi(S) \subset S\} \subset \mathrm{L}(V)$.

Proposition 3.4. $\mathrm{L}(V)_S = \{\sigma \in \mathrm{End}(A(V, S)) \mid \sigma(V) \subset V\}$.

Proof. By definition, the elements of $A(V, S)$ are the equivalence classes $x + I(S)$, $x \in \mathrm{T}(V)_+$. Let us show the inclusion "$\supset$". Consider $\sigma \in \mathrm{End}(A(V, S))$ such that $\sigma(V) \subset V$. The σ-action coincides with the action of $\widetilde{\sigma} := \sigma|_V \in \mathrm{L}(V)$ on $A(V, S)$ by virtue of (3.2), since the algebra $A(V, S)$ is generated by V. The σ-action preserves zero of $A(V, S)$, hence $\widetilde{\sigma}(I(S)) \subset I(S)$ and $\sigma \in \mathrm{L}(V)_S$.

Now we turn to the inverse inclusion. For arbitrary subsets $X, Y \subset \mathrm{T}(V)$ define

$$X \otimes Y := \{x \otimes y \mid x \in X, y \in Y\} \subset \mathrm{T}(V).$$

Let $\sigma \in \mathrm{L}(V)_S$. Then

$$\sigma((x + I(S)) \otimes (y + I(S))) \subset \sigma(x \otimes y) + I(S) = \sigma(x) \otimes \sigma(y) + I(S)$$

by definition of the $\mathrm{L}(V)$-action on $\mathrm{T}(V)_+$. Hence $\sigma \in \mathrm{End}(A(V, S))$. $\square$

3.2.2 Algebra $D(P, U, S, \gamma)$

Lemma 3.5. *Let A be an algebra with a left identity $e \in A$ such that* $\mathrm{vect}(A) = \langle e\rangle \oplus A_1 \oplus \cdots \oplus A_r$, *where A_i is the eigenspace with an eigenvalue $\alpha_i \neq 0, 1$ for the operator of right multiplication of A by e. Assume that 0 and e are the only idempotents in A. Then*

1. *e is the unique left identity in A;*
2. *if $\sigma \in \mathrm{End}(A)$, then either $\sigma(e) = e$ and $\sigma(A_i) \subset A_i$ for any i, or $\sigma = \mathfrak{z}$.*

Proof. (i) The left identity is a nonzero idempotent. Hence it is unique.

(ii) Since the image of an idempotent is an idempotent, $\sigma(e) = 0$ or $\sigma(e) = e$. If $\sigma(e) = 0$, then $\sigma(a) = \sigma(ea) = \sigma(e)\sigma(a) = 0$, i.e. $\sigma = \mathfrak{z}$. Now assume that $\sigma(e) = e$. Then $\sigma(A_i)$ is the eigenspace with an eigenvalue $\alpha_i \neq 0, 1$ for the operator of right multiplication by e. Hence $\sigma(A_i) \subset A_i$. □

Let P be a two-dimensional vector space with a basis $\{p_1, p_2\}$, U be a nonzero finite-dimensional space, and

$$V := P \oplus U. \tag{3.6}$$

We fix an integer $r > 1$ and also

(i) a subspace $S \subset V^{\otimes r}$;

(ii) a sequence $\gamma = (\gamma_1, \ldots, \gamma_6) \in (\mathbb{K} \setminus \{0, 1\})^6$, where $\gamma_i \neq \gamma_j$ for $i \neq j$.

Define an algebra extension $D(P, U, S, \gamma) \supset A(V, S)$ in the following way. Let $b, c, d, e \in D(P, U, S, \gamma)$ be such that

$$\mathrm{vect}(D(P, U, S, \gamma)) = \langle e\rangle \oplus \langle b\rangle \oplus \langle c\rangle \oplus \langle d\rangle \oplus \mathrm{vect}(A(V, S)) \tag{3.7}$$

and the following conditions hold:

(D1) e is the left identity of $D(P, U, S, \gamma)$;

(D2) $\langle b\rangle$, $\langle c\rangle$, $\langle d\rangle$ as well as $P, U \subset V = V^{\otimes 1} \subset A(V, S)$ and $(\bigoplus_{i=2}^{r-1} V^{\otimes i}) \oplus (V^{\otimes r}/S) \subset A(V, S)$ are the eigenspaces with the eigenvalues $\gamma_1, \ldots, \gamma_6$ respectively of the operator of right multiplication by e;

(D3) The multiplication table for b, c, d is

$$\begin{array}{lll} b \cdot b := 0, & b \cdot c := c + \gamma_{bc} b, & b \cdot d := 0, \\ c \cdot b := -c, & c \cdot c := b, & c \cdot d := e, \\ d \cdot b := p_1, & d \cdot c := d, & d \cdot d := p_2, \end{array} \tag{3.8}$$

where $\gamma_{bc} = \frac{\gamma_2 - \gamma_1}{\gamma_2 - \gamma_3}$;

(D4) $\langle b,c,d\rangle \cdot A(V,S) = A(V,S)\cdot\langle b,c,d\rangle = 0$.

Define the action of $g \in \mathrm{L}(V)_S$ on $\mathrm{vect}(D(P,U,S,\gamma))$ as follows: $g|_{\langle b\rangle} = g|_{\langle c\rangle} = g|_{\langle d\rangle} = g|_{\langle e\rangle} = \mathrm{id}$, $g|_V$ is the natural $\mathrm{L}(V)$-action on V, and on the other summands of $A(V,S)$ it is defined by (3.2). By Proposition 3.4 we may identify $\mathrm{L}(V)_S$ with the corresponding submonoid of $\mathrm{L}(\mathrm{vect}(D(P,U,S,\gamma)))$. Further, we may consider an embedding $\mathrm{L}(U) \hookrightarrow \mathrm{L}(V)$, $h \mapsto \mathrm{id}|_P \oplus h$. Thus, $\mathrm{L}(U)_S \subset \mathrm{L}(V)_S$, and we obtain $\mathrm{L}(U)_S$-action on $\mathrm{vect}(D(P,U,S,\gamma))$.

Proposition 3.6. *We have*

$$\mathrm{End}(D(P,U,S,\gamma)) = \mathrm{L}(U)_S \sqcup \{\mathfrak{z}\},$$

where $\{\mathfrak{z}\}$ *is an (isolated) component of the monoid* $\mathrm{End}(D(P,U,S,\gamma))$.

Proof. First of all, we show that 0 and e are the only idempotents of $D(P,U,S,\gamma)$. Indeed, let $\varepsilon = \lambda_e e + \lambda_b b + \lambda_c c + \lambda_d d + a$, where $a \in A(V,S)$. Then

$$\begin{aligned}\varepsilon^2 = (\lambda_e^2 + \lambda_c\lambda_d)e + (\lambda_b\lambda_e(1+\gamma_1) + \lambda_c^2 + \lambda_b\lambda_c\gamma_{bc})b +\\ + \lambda_c\lambda_e(1+\gamma_2)c + ((1+\gamma_3)\lambda_d\lambda_e + \lambda_d\lambda_c)d + a' =\\ = \lambda_1 e + \lambda_2 d + \lambda_3 c + \lambda_4 b + a, \text{ where } a, a' \in A(V,S).\end{aligned} \tag{3.9}$$

Hence

$$\lambda_e = \lambda_e^2 + \lambda_c\lambda_d, \tag{3.10}$$
$$\lambda_b = \lambda_b\lambda_e(1+\gamma_1) + \lambda_c^2 + \lambda_b\lambda_c\gamma_{bc}, \tag{3.11}$$
$$\lambda_c = \lambda_c\lambda_e(1+\gamma_2), \tag{3.12}$$
$$\lambda_d = \lambda_d\lambda_e(1+\gamma_3) + \lambda_c\lambda_d. \tag{3.13}$$

Assume $\lambda_c \neq 0$. By (3.12)

$$1+\gamma_2 \neq 0, \lambda_e = \frac{1}{1+\gamma_2} \quad \text{and} \quad \lambda_c\lambda_d = \lambda_e - \lambda_e^2 \neq 0,$$

so $\lambda_d \neq 0$. Hence equation (3.13) implies

$$\lambda_c = 1 - \lambda_e(1+\gamma_3) = \frac{\gamma_2-\gamma_3}{1+\gamma_2}$$

. Finally, by (3.11) we have

$$\lambda_c^2 = \lambda_b(1 - \lambda_e(1+\gamma_1) - \lambda_c\gamma_{bc}) = \lambda_b\left(\frac{\gamma_2-\gamma_1}{1+\gamma_2} - \frac{\gamma_2-\gamma_3}{1+\gamma_2}\cdot\frac{\gamma_2-\gamma_1}{\gamma_2-\gamma_3}\right) = 0.$$

From this contradiction we deduce $\lambda_c = 0$.

Moreover, $\lambda_e = 0$ or $\lambda_e = 1$ by (3.10). If $\lambda_e = 0$, then $\lambda_b = \lambda_d = 0$, $\varepsilon = a \in A(V,S)$ and $\varepsilon = 0$, since zero is the only idempotent of $A(V,S)$. Now assume $\lambda_e = 1$. From equations (3.11) and (3.13) accordingly follow $\lambda_b = 0$ and $\lambda_d = 0$. Thus, $\varepsilon = e + a$, $a \in A(V,S)$.

Let $a = a_P + a_U + a_\Sigma$, where $a_P \in P, a_U \in U, a_\Sigma \in (\bigoplus_{i=2}^{r-1} V^{\otimes i}) \oplus (V^{\otimes r}/S)$. Then

$$\varepsilon^2 = e + (1+\gamma_4)a_P + (1+\gamma_5)a_U + a'_\Sigma = e + a_P + a_U + a_\Sigma, \tag{3.14}$$

where $a'_\Sigma \in (\bigoplus_{i=2}^{r-1} V^{\otimes i}) \oplus (V^{\otimes r}/S)$. Hence $a_U = a_P = 0$. Assume $a_\Sigma \neq 0$, then we may write $a_\Sigma = a_k + \ldots + a_r$, $a_k \neq 0$, where $a_i \in V^{\otimes i}$ for $i < r$ and $a_r \in V^{\otimes r}/S$. This way,

$$(e + a_k + \ldots + a_r)^2 = e + (1+\gamma_6)a_k + a'' = e + a_k + \ldots + a_r, \tag{3.15}$$

where $a'' \in (\bigoplus_{i=k+1}^{r-1} V^{\otimes i}) \oplus (V^{\otimes r}/S)$ for $k < r$ and $a'' = 0$ for $k = r$. This implies $a_k = 0$, which is a contradiction. Hence $a_\Sigma = 0$ and $\varepsilon = e$.

Thus, $D(P,U,S,\gamma)$ contains no idempotent different from 0 and e. Let $\sigma \in \mathrm{End}(D(P,U,S,\gamma)) \setminus \{\mathfrak{z}\}$. By Lemma 3.5 $\sigma(e) = e$ and $\langle b\rangle$, $\langle c\rangle$, $\langle d\rangle$, P, U, $A(V,S)$ are σ-invariant. Let $\sigma(b) = \delta_b b$, $\sigma(c) = \delta_c c$, $\sigma(d) = \delta_d d$. The equations $cd = e$, $dc = d$, $cb = -c$ imply $\delta_c\delta_d = 1$, $\delta_c\delta_d = \delta_d$, $\delta_b\delta_c = \delta_c$. One may check that $\delta_b = \delta_c = \delta_d = 1$. Finally, the equations $db = p_1$, $dd = p_2$ imply $\sigma|_P = \mathrm{id}_P$.

Since V and $A(V,S)$ are σ-invariant, $\sigma|_{A(V,S)} \in \mathrm{L}(V)_S$ by Proposition 3.4. Taking into account that $\sigma|_P = \mathrm{id}_P$ and $\sigma(U) \subset U$, we obtain $\sigma \in \mathrm{L}(U)_S$. □

3.3 Affine monoids as the normalizers of linear subspaces

Proposition 3.7. *Let M be an affine algebraic monoid. There is a finite-dimensional vector space U and an integer $r > 1$ such that the following holds. Let P be a two-dimensional vector space with a trivial $\mathrm{L}(U)$-action. Then the $\mathrm{L}(U)$-module $(P \oplus U)^{\otimes r}$ contains a linear subspace S such that $\mathrm{L}(U)_S \cong M$.*

Proof. Since there exists a closed embedding $M \hookrightarrow \mathrm{L}(U)$ for some finite-dimensional space U, we may suppose $M \subset \mathrm{L}(U)$. Consider the action of $\mathrm{L}(U)$ on itself by left multiplication. In addition, we consider the regular representation of $\mathrm{L}(U)$ on the algebra $\mathbb{K}[\mathrm{L}(U)]$,

$$(g \cdot f)(u) := f(ug), \quad g, u \in \mathrm{L}(U), f \in \mathbb{K}[\mathrm{L}(U)]. \tag{3.16}$$

Denote $d := \dim U$. Note that the $\mathrm{L}(U)$-modules $\mathbb{K}[\mathrm{L}(U)]$ and $\mathrm{Sym}(U^{\oplus d})$ are isomorphic. To prove this, it suffices to associate a linear function on $\mathrm{L}(U)$ to every vector $(u_1, \dots, u_d) \in U^{\oplus d}$, since $\mathbb{K}[\mathrm{L}(U)] = \mathrm{Sym}(\mathrm{L}(U)^*)$. Identify U with $\mathbb{K}^d$, $\mathrm{L}(U)$ with $\mathrm{Mat}_{d\times d}(\mathbb{K})$. Let A be in $\mathrm{L}(U)$, B be a matrix with columns $u_1, \dots, u_d$. Let $l_{u_1,\dots,u_d}(A) := \mathrm{tr}\, AB$. Then $(g \cdot l_{u_1,\dots,u_d})(A) = \mathrm{tr}\, AgB = l_{gu_1,\dots,gu_d}(A)$, i.e. we have an $\mathrm{L}(U)$-equivariant isomorphism.

By definition of symmetric algebra fix a natural epimorphism

$$\xi\colon \mathrm{T}(U^{\oplus d}) \to \mathrm{Sym}(U^{\oplus d}) \cong \mathbb{K}[\mathrm{L}(U)]. \tag{3.17}$$

There is a finite-dimensional subspace $W \subset \mathbb{K}[\mathrm{L}(U)]$ such that

$$\mathrm{L}(U)_W = M. \tag{3.18}$$

Let us call a linear span of a set $\{L(U) \cdot f\}$ an *$L(U)$-spread* of f. In order to prove this, one may show that an $\mathrm{L}(U)$-spread of an arbitrary function $f \in \mathbb{K}[\mathrm{L}(U)]$ is finite-dimensional. Indeed, since the $\mathrm{L}(U)$-action is a morphism, $(g \cdot f)(u) = f(ug) \in \mathbb{K}[\mathrm{L}(U) \times \mathrm{L}(U)] = \mathbb{K}[\mathrm{L}(U)] \otimes \mathbb{K}[\mathrm{L}(U)]$, where $u, g \in \mathrm{L}(U)$, and there are functions $F_j, H_j \in \mathbb{K}[\mathrm{L}(U)]$ such that

$$(g \cdot f)(u) = \sum_{j=1}^{n} F_j(u) H_j(g). \tag{3.19}$$

Therefore, the $\mathrm{L}(U)$-spread of the function f is contained in the finite-dimensional subspace $\langle F_1, \dots, F_n \rangle$.

Let $I(M) = (f_1, \dots, f_t) \lhd \mathbb{K}[\mathrm{L}(U)]$ be the ideal of functions vanishing on M. Summing the $\mathrm{L}(U)$-spreads of the functions f_i we obtain a finite-dimensional $\mathrm{L}(U)$-invariant subspace $V \subset \mathbb{K}[\mathrm{L}(U)]$. Define $W = I(M) \cap V$. First, it contains $f_1, \dots, f_t$. Second, it is M-invariant, since the ideal $I(M)$ is M-invariant. Obviously, $g \in M$ implies $g \cdot W \subset W$. On the other hand, suppose that $g \cdot W \subset W$, where $g \in \mathrm{L}(U)$. Then $f_i(g) = (g \cdot f_i)(E) = 0$ for $i = 1, \dots, t$, where E is the identity of $\mathrm{L}(U)$ and is automatically contained in M. Therefore, $g \in M$. This proves (3.18).

Furthermore, since the space W is finite-dimensional, there is an integer $h \in \mathbb{Z}_+$ such that

$$W \subset \xi(\textstyle\bigoplus_{i\le h} (U^{\oplus d})^{\otimes i}). \tag{3.20}$$

Define $W' := \xi^{-1}(W) \cap (\bigoplus_{i\le h} (U^{\oplus d})^{\otimes i})$. The $\mathrm{L}(U)$-equivariance of ξ implies

$$\mathrm{L}(U)_{W'} = \mathrm{L}(U)_W. \tag{3.21}$$

Fix a basis $\{p_1, p_2\}$ of the space P. There exists an embedding of $\mathrm{L}(U)$-modules

$$\iota\colon \mathrm{T}(U^{\oplus d}) \hookrightarrow \mathrm{T}(\langle p_1 \rangle \oplus U). \tag{3.22}$$

Indeed, let U_i be the ith summand of $U^{\oplus d}$. Consider an arbitrary basis $\{f_{ij} \mid j = 1, \ldots, d\}$ of U_i and define an embedding as follows,

$$\iota(f_{i_1 j_1} \otimes \ldots \otimes f_{i_t j_t}) := p_1^{\otimes i_1} \otimes f'_{i_1 j_1} \otimes \ldots \otimes p_1^{\otimes i_t} \otimes f'_{i_t j_t}, \tag{3.23}$$

where f'_{ij} is the image of f_{ij} under the identity isomorphism $U_i \to U$. It is easy to check that the embedding $\iota\colon \mathrm{T}(U^{\oplus d}) \hookrightarrow \mathrm{T}(\langle p_1 \rangle \oplus U)$ defined on the basis of $\mathrm{T}(U^{\oplus d})_+$ by formula (3.23) and sending 1 to 1 is as required.

Now we may consider a space $W'' := \iota(W')$,

$$\mathrm{L}(U)_{W''} = \mathrm{L}(U)_{W'}. \tag{3.24}$$

Since W'' is finite-dimensional, there exists an integer $b \in \mathbb{N}$ such that

$$W'' \subset \bigoplus_{i \le b} (\langle p_1 \rangle \oplus U)^{\otimes i}. \tag{3.25}$$

Take $r \ge b$ such that $r > 1$ and consider a linear mapping

$$\iota_r\colon \bigoplus_{i \le b} (\langle p_1 \rangle \oplus U)^{\otimes i} \to (P \oplus U)^{\otimes r}, \quad f_i \mapsto p_2^{\otimes (r-i)} \otimes f_i, f_i \in (\langle p_1 \rangle \oplus U)^{\otimes i}. \tag{3.26}$$

Obviously, ι_r is an embedding of $\mathrm{L}(U)$-modules. Define $S = \iota_r(W'')$. Then

$$\mathrm{L}(U)_S = \mathrm{L}(U)_{W''}. \tag{3.27}$$

Now the claim follows from equations (3.18), (3.21), (3.24), and (3.27). $\square$

Proof of Theorem 3.1

Let M be an arbitrary affine algebraic monoid, U, b, r, P, S be as in Proposition 3.7. Fix a set $\gamma \in (\mathbb{K} \setminus \{0, 1\})^6$ such that $\gamma_i \neq \gamma_j$ for $i \neq j$, and consider the algebra $D(P, U, S, \gamma)$. It follows from Proposition 3.7 and Proposition 3.6 that $\mathrm{End}(D(P, U, S, \gamma)) \cong M \sqcup \{\mathfrak{z}\}$. This ends the proof. $\square$

Chapter 4

Solvability of automorphism groups

4.1 Introduction

In this chapter the field $\mathbb{K}$ is assumed to be algebraically closed of characteristic zero. We denote by R the algebra of formal power series $\mathbb{K}[\![x_1, \ldots, x_n]\!]$ and by $\mathfrak{m}$ the maximal ideal $(x_1, \ldots, x_n) \lhd R$. Let $I \subset \mathfrak{m}$ be such that $S = R/I$ is a finite-dimensional (or Artin) local algebra with the maximal ideal $\bar{\mathfrak{m}} = \mathfrak{m}/I$.

Consider the automorphism group $\operatorname{Aut} S$. This is an affine algebraic group with tangent algebra being the Lie algebra of derivations $\operatorname{Der} S$; see [13, Ch.1, §2.3, ex. 2]. Thus the solvability of the connected component of identity $\operatorname{Aut}^\circ S$ (or the *identity component* for short) is equivalent to the solvability of the Lie algebra $\operatorname{Der} S$.

In 2009 M. Schulze obtained the following criterion, which has several applications discussed below.

Theorem 4.1 (Schulze, [19])**.** *Let $S = R/I$ be a finite-dimensional local algebra, where $I \subset \mathfrak{m}^l$. If the inequality*

$$\dim(I/\mathfrak{m}I) < n + l - 1 \tag{4.1}$$

holds, then the algebra of derivations $\operatorname{Der} S$ *is solvable.*

Hereafter we provide a generalization of that criterion to a non-local case, see Corollary 4.26, as well as establishing a new criterion based on similar techniques, see Theorem 4.12. These two criteria work for different types of algebras.

In order to mention some applications of Schulze's criterion, let us consider a *regular sequence* $f_1, \ldots, f_n \in \mathbb{K}[x_1, \ldots, x_n]$[1]. Equivalently, the quotient $S = \mathbb{K}[x_1, \ldots, x_n]/(f_1, \ldots, f_n)$ is non-trivial and finite-dimensional and is called a *global complete intersection.*

[1]i.e. for all i the image of f_i in the quotient $\mathbb{K}[x_1, \ldots, x_n]/(f_1, \ldots, f_{i-1})$ is not a zero divisor.

Conjecture 4.2 (Halperin, 1987[2]). *Suppose that a finite-dimensional algebra S is a global complete intersection. Then the identity component* $\operatorname{Aut}^\circ S$ *of the automorphism group of S is solvable.*

Just a few months later H. Kraft and C. Procesi proved the conjecture in the case of homogeneous polynomials.

Theorem 4.3 (Kraft–Procesi, [11]). *Assuming $\mathbb{K} = \mathbb{C}$, let $f_1, \dots, f_n \in \mathbb{K}[x_1, \dots, x_n]$ be homogeneous polynomials, and the algebra*

$$S = \mathbb{K}[x_1, \dots, x_n]/(f_1, \dots, f_n) \tag{4.2}$$

be finite-dimensional. Then the identity component $\operatorname{Aut}^\circ S$ *is solvable.*

Note that in terms of Proposition 4.3 the sequence $f_1, \dots, f_n$ is regular. In this case the algebra S is local. Thus, the following generalization of Theorem 4.3 turns out to be a direct consequence of Schulze's Theorem 4.1.

Corollary 4.4 (Schulze, [19, Corollary 2]). *Given a local complete intersection $S = R/(f_1, \dots, f_n)$, the group* $\operatorname{Aut}^\circ S$ *is solvable.*

Proof. We may assume that $f_i \in \mathfrak{m}^2$. Then $\dim(I/\mathfrak{m}I) = n$ and so we can apply (4.1). $\square$

In Section 4.4 we introduce a criterion for the solvability of the algebra of derivations $\operatorname{Der} S$ for a non-local finite-dimensional algebra S, see Theorem 4.25. This allows us to deduce the global case of Conjecture 4.2 from the local one, and so to finish the proof.

Let us consider now isolated hypersurface singularities (or IHS, for short). Let $p \in \mathbb{K}[x_1, \dots, x_n]$ be such that the hypersurface $\{p = 0\} \subset \mathbb{K}^n$ has an isolated singularity $H = (\{p = 0\}, 0)$ at the origin. Let $J(p) = \left\langle \frac{\partial p}{\partial x_1}, \dots, \frac{\partial p}{\partial x_n} \right\rangle_{\mathbb{K}}$ be the *Jacobian* of p. So, the ideal $(J(p))$ is the *Jacobian ideal* of p. The quotient $A(H) = \mathbb{K}[\![x_1, \dots, x_n]\!]/(p, J(p))$ is called a *local algebra* or a *moduli algebra* of the IHS H. It is also known as *Tyurina algebra* in singularity theory.

Since the formal power series ring is local, the algebra $A(H)$ is local as well. There is an analogue of the Hilbert Nullstellensatz for the germs of analytic functions (called *Rückert Nullstellensatz*, see [9, Theorem 3.4.4], [1, 30.12], [21]), which holds for formal power series as well, since there is a purely algebraic proof. Hence the radical $\sqrt{(p, J(p))}$ coincides with the maximal ideal $\mathfrak{m}$. Thus the ideal $(p, J(p))$ contains some power of the maximal ideal or, equivalently, the algebra $A(H)$ is finite-dimensional. Conversely, if the algebra $A(H)$ is finite-dimensional then the

[2] This conjecture was proposed by S. Halperin at the conference in honor of J.–L. Koszul.

singularity H is isolated. Indeed, the finite dimensionality of A is equivalent to an inclusion $\mathfrak{m}^r \subset I$ for some r, or $\sqrt{(p, J(p))} = \mathfrak{m}$. This implies $\mathbb{V}(p, J(p)) = 0$, and H is the only singularity in some neighbourhood of zero.

It has been proven by J. Mather and S. S.–T. Yau in [12] that two IHS are biholomorphically equivalent if and only if their moduli algebras are isomorphic. Thus, the finite-dimensional local algebra $A(H)$ defines the IHS H up to analytic isomorphism.

In order to determine as to when a finite-dimensional local algebra is a moduli algebra of some IHS, S.S.–T. Yau [23] introduced a Lie algebra of derivations $L(H) = \operatorname{Der} A(H)$ called sometimes a *Yau algebra*. He obtained the following result.

Theorem 4.5 (S.S.–T. Yau [24])**.** *The algebra $L(H)$ of an IHS H is solvable.*

Note that generally the Yau algebra does not uniquely determine its moduli algebra. But for *simple* singularities this property holds with only one exception. Their classification is well known and consists of two infinite series A_k, D_k and three exceptional singularities E_6, E_7, E_8; e.g. see [2, Chapter 2]. A. Elashvili and G. Khimshiashvili proved the following fact.

Theorem 4.6 (Elashvili–Khimshiashvili, [7, Theorem 3.1])**.** *Let H_1 and H_2 be two simple IHS, except the pair A_6 and D_5. Then $L(H_1) \cong L(H_2)$ if and only if H_1 and H_2 are analytically isomorphic.*

Remark 4.7. Assume that the polynomial p is quasi-homogeneous, i.e.

$$p(\lambda^{k_1} x_1, \dots, \lambda^{k_n} x_n) = \lambda^k p(x_1, \dots, x_n) \text{ for some fixed } k, k_1, \dots, k_n \in \mathbb{N}. \tag{4.3}$$

Then $p \in J(p)$ and the moduli algebra $\mathbb{K}[\![x_1, \dots, x_n]\!]/(p, J(p))$ is a complete intersection. Under this assumption Theorem 4.5 is a particular case of Corollary 4.4.

In [19] M. Schulze deduces Theorem 4.5 from his criterion. In order to prove it he uses the following deep result of G. Kempf.

Theorem 4.8 (Kempf, [10, Theorem 13])**.** *Let p be a homogeneous polynomial of degree $d \geq 3$ defined as a regular function on the space $\mathbb{C}^n$ endowed with a linear action of a semisimple group G. If the Jacobian $J(p)$ is a G-invariant subspace then there exists a G-invariant polynomial q such that $J(p) = J(q)$.*

Definition 4.9. We say that the finite-dimensional local algebra S is an *extremal algebra* if the equality $\dim I/\mathfrak{m}I = l + n - 1$ holds.

The specification of the extremal algebras with a non-solvable algebra of derivations allows to deduce Theorem 4.5 from Schulze's criterion in a direct way, as explained in Section 4.3.

Definition 4.10. For a graded ideal J let us denote by J_k its kth graded component. We say that a graded local finite-dimensional algebra $S = R/I$ is *narrow* if the inequality

$$\dim I_k - \dim(\mathfrak{m}I)_k \leq k \tag{4.4}$$

holds for all $k = 1, 2 \ldots$. In other words, there exists a set of homogeneous generators of I such that the number of generators of degree k is not greater than k for each k.

Remark 4.11. Let $I \supset \mathfrak{m}^r$. Then the algebra $S = R/I$ is narrow if the inequality (4.4) holds for all $k \leq r$. Indeed, $I_k = (\mathfrak{m}I)_k$ for $k > r$.

Recall that the *associated graded algebra* of the local algebra S is the algebra

$$\operatorname{gr} S = \mathbb{K} \oplus (\bar{\mathfrak{m}}/\bar{\mathfrak{m}}^2) \oplus (\bar{\mathfrak{m}}^2/\bar{\mathfrak{m}}^3) \oplus \ldots$$

, i.e. $(\operatorname{gr} S)_i = \bar{\mathfrak{m}}^i/\bar{\mathfrak{m}}^{i+1}$. Our solvability criterion is as follows.

Theorem 4.12. *Suppose that the associated graded algebra* $\operatorname{gr} S$ *of a local finite-dimensional algebra S is narrow. Then the algebra of derivations* $\operatorname{Der} S$ *is solvable.*

The proof is given in the next section. In the last section we give a lower bound for the dimension of the automorphism group and obtain an algebra with a unipotent automorphism group.

Let us mention a related result on the solvability of the group of equivariant automorphisms. Consider a connected affine algebraic group G and an irreducible affine G-variety X. Assume that the number of G-orbits on X is finite and X contains a G-fixed point. Then the identity component $\operatorname{Aut}_G^\circ X$ of the group of G-equivariant automorphisms of the variety X is solvable; see [4, Theorem 1].

4.2 Solvability criteria

In this section we provide a simplified proof of Theorem 4.1 and introduce a new solvability criterion.

If $I \supset \mathfrak{m}^r \lhd \mathbb{K}[x_1, \ldots, x_n]$ for some r then $R/\widetilde{I} \cong \mathbb{K}[x_1, \ldots, x_n]/I$, where $\widetilde{I}$ is the ideal in the algebra of formal power series generated by I. Therefore it makes no difference whether the local algebra is obtained by factorizing the algebra of polynomials or the algebra of formal power series.

Proposition 4.13. *Suppose that the ideal $I \lhd R$ is represented in the form $I = W \oplus \mathfrak{m}I$. Then the ideal (W) generated by the subspace W coincides with I.*

Proof. Consider the factorization mapping $\varphi\colon R \to R/(W)$. The quotient algebra is a local algebra with maximal ideal $\varphi(\mathfrak{m})$. The decomposition $I = W \oplus \mathfrak{m}I$ implies $\varphi(I) = \varphi(\mathfrak{m}I)$. Since the ring R is Noetherian, the ideals I and $\varphi(I)$ are finitely generated. Then by Nakayama's Lemma (see [5, Proposition 2.6]) we have $\varphi(I) = 0$, i.e. $(W) = I$. $\square$

Corollary 4.14. *The minimal number of generators of the ideal I is equal to* $\dim W = \dim(I/\mathfrak{m}I)$.

Note that Proposition 4.13 does not hold for the algebra of polynomials. For example take an ideal $I = \mathfrak{m}^2 \lhd \mathbb{K}[x]$ and decompose it as follows,

$$\mathfrak{m}^2 = \langle x^2 - x^3 \rangle \oplus \mathfrak{m}^3. \tag{4.5}$$

It is easy to see that the ideal $(x^2 - x^3)$ does not coincide with $\mathfrak{m}^2$.

Below we follow the ideas of [19].

As always we suppose that $S = R/I$, where $\mathfrak{m}^l \supset I \lhd R = \mathbb{K}[\![x_1, \dots, x_n]\!]$ for $l \geq 2$. Assume that the algebra $\operatorname{Der} S$ is not solvable. Hence it contains an $\mathfrak{sl}_2$-triple $\{e, f, h\}$ with relations $[e, f] = h$, $[h, f] = -2f$, $[h, e] = 2e$. Note that the automorphisms of S preserve the maximal ideal $\bar{\mathfrak{m}} \lhd S$ as it is unique, and all powers of $\bar{\mathfrak{m}}$ as well. Therefore the ideals $\bar{\mathfrak{m}}$ and $\bar{\mathfrak{m}}^2$ are $\mathfrak{sl}_2$-submodules. Since the representations of $\mathfrak{sl}_2$ are completely reducible, the ideal $\bar{\mathfrak{m}}$ contains an $\mathfrak{sl}_2$-submodule $\overline{V}$ such that

$$\bar{\mathfrak{m}} = \overline{V} \oplus \bar{\mathfrak{m}}^2. \tag{4.6}$$

Denote by $\varphi\colon R \to S$ the factorization by the ideal I. Since $\varphi(\mathfrak{m}^2) = \bar{\mathfrak{m}}^2$ there exists a subspace $V \subset \mathfrak{m}$ such that $\mathfrak{m} = V \oplus \mathfrak{m}^2$ and $\varphi\colon V \xrightarrow{\sim} \overline{V}$. Thus, according to Proposition 4.13 the subspace V generates the ideal $\mathfrak{m}$ as an algebra. Hence it generates also the algebra R. So up to a change of coordinates we may assume that $V = \langle x_1, \dots, x_n \rangle$ and $\overline{V} = \langle \bar{x}_1, \dots, \bar{x}_n \rangle$, where $\bar{x}_i = \varphi(x_i)$.

We may introduce an $\mathfrak{sl}_2$-representation on V by the given isomorphism and extend it to R. Note that the factorization map φ is a homomorphism of $\mathfrak{sl}_2$-modules. Therefore the ideal $I \lhd R$ is an invariant subspace of the $\mathfrak{sl}_2$-representation on R.

Given a *weight vector* z, i.e. an eigenvector of the operator $h \in \mathfrak{sl}_2$, denote its weight by $\operatorname{wt}(z) \in \mathbb{Z}$. We may suppose that $x_1, \dots, x_n$ are the weight vectors of the $\mathfrak{sl}_2$-module V, and $x_1, \dots, x_k$, $k \leq n$, are the highest weight vectors with weights $n_i = \operatorname{wt}(x_i)$, where $n_1 \geq \dots \geq n_k \geq 0$, $\sum(n_i + 1) = n$. Denote $V_{high} := \langle x_1, \dots, x_k \rangle$, $V_{rest} := \langle x_{k+1}, \dots, x_n \rangle$.

The ideal $\mathfrak{m}I \subset R$ is $\mathfrak{sl}_2$-invariant by the Leibniz rule, hence I contains the complementary $\mathfrak{sl}_2$-submodule W such that $I = W \oplus \mathfrak{m}I$. By Corollary 4.14 its basis is a minimal set generating I.

Similarly to $V = V_{high} \oplus V_{rest}$ consider the decomposition

$$W = W_{high} \oplus W_{rest} \tag{4.7}$$

into the subspace $W_{high} = \langle w_1, \ldots, w_s \rangle$, where w_i are the highest weight vectors of W, and the subspace W_{rest} of the remaining weight vectors of W. Notice that $W_{rest} \subset \operatorname{Im} f \subset (x_{k+1}, \ldots, x_n)$ since $\operatorname{Im} f$ is spanned by the weight vectors which are not of highest weight.

Let $\varphi_i \colon R \to R/J_i$ be the factorization by the ideal $J_i = (x_{i+1}, \ldots, x_n)$, $\quad i = 1 \ldots, k$. Since $J_i \supset (x_{k+1}, \ldots, x_n) \supset W_{rest}$ the equality $W_i := \varphi_i(W) = \varphi_i(W_{high})$ holds. Note that $\dim W_i \geq i$, because $\mathbb{K}[\![x_1, \ldots, x_i]\!]/(W_i) \cong R/(J_i, W_i) \cong S/(\bar{x}_{i+1}, \ldots, \bar{x}_n)$ is finite-dimensional. In particular, $s \geq k$.

By induction we can reorder the highest weight vectors $w_1, \ldots, w_s \in W_{high}$ so that $\varphi_i(w_1), \ldots, \varphi_i(w_i)$ become linearly independent in W_i for all i. Then $\operatorname{wt}(w_i) \geq ln_i$ since w_i contains the monomials in variables $\bar{x}_1, \ldots, \bar{x}_i$ of degree at least l and $\operatorname{wt}(x_j) = n_j \geq n_i$ for $j \leq i$.

Proof of Theorem 4.1. The above discussion implies that the subspace V contains a non-trivial $\mathfrak{sl}_2$-submodule and that $n_1 > 0$. We have

$$\begin{aligned} \dim I/\mathfrak{m}I = \dim W \geq \sum_{i=1}^{k}(ln_i + 1) = (n_1 - 1)l + l + 1 + \sum_{i=2}^{k}(ln_i + 1) \geq \\ (n_1 - 1) + l + 1 + \sum_{i=2}^{k}(n_i + 1) = \sum_{i=1}^{k}(n_i + 1) + l - 1 = n + l - 1, \end{aligned} \tag{4.8}$$

as required. □

Proposition 4.15. *There exists a natural mapping* $\varphi \colon \operatorname{Aut} S \to \operatorname{Aut}(\operatorname{gr} S)$ *with a unipotent kernel.*

Proof. The ideals $\bar{\mathfrak{m}}^i$ are invariant under $\operatorname{Aut} S$ for all i, since they are powers of the unique maximal ideal. Therefore $\operatorname{Aut} S$ acts naturally on $\bar{\mathfrak{m}}^i/\bar{\mathfrak{m}}^{i+1}$ for all i, hence it acts on $\operatorname{gr} S$. We obtain a natural map $\varphi \colon \operatorname{Aut} S \to \operatorname{Aut}(\operatorname{gr} S)$.

Choose a basis of S which is consistent with the chain of subspaces $0 \subset \bar{\mathfrak{m}}^r \subset \ldots \subset \bar{\mathfrak{m}} \subset S$. Consider an arbitrary operator $g \in \ker \varphi$. Then $g(z) \in z + \bar{\mathfrak{m}}^{i+1}$ for any $z \in \bar{\mathfrak{m}}^i$, and g is represented by a unitriangular matrix in the chosen basis. Hence $\ker \varphi$ is unipotent. □

Corollary 4.16. *If the identity component* $\operatorname{Aut}^\circ(\operatorname{gr} S)$ *is solvable then the identity component* $\operatorname{Aut}^\circ S$ *is solvable as well.*

Theorem 4.17. *The algebra of derivations* $\operatorname{Der} S$ *of a narrow algebra* S *is solvable.*

Proof. Let $\operatorname{Der} S$ be non-solvable. Then the discussion above is applicable. Consider a simple $\mathfrak{sl}_2$-submodule $F = \mathfrak{sl}_2 \cdot w_1 \subset I$. It has zero intersection with the ideal $\mathfrak{m}I$. Let k be the largest integer such that $F \subset \mathfrak{m}^k$. Then F has zero intersection with $\mathfrak{m}^{k+1}$ as well and the highest weight of F is equal to $kn_1 \geq k$. After factorization by $\mathfrak{m}^{k+1}$ this implies that $\dim I_k \geq \dim(\mathfrak{m}I)_k + \dim F > \dim(\mathfrak{m}I)_k + k$, and thus the algebra S is not narrow. $\square$

Proof of Theorem 4.12. The desired statement is a direct consequence of Theorem 4.17 and Corollary 4.16. $\square$

Remark 4.18. In fact, Proposition 4.15 is not necessary for the proof of Theorem 4.12, but makes it more clear.

Example 4.19. The algebras

$$A = \mathbb{K}[x,y]/(x^2, y^3, xy^2), \tag{4.9}$$
$$B = \mathbb{K}[x,y,z]/(x^3, x^2y, x^2z, y^4, z^4) \tag{4.10}$$

are extremal and Schulze's criterion does not apply, but their algebras of derivations are solvable due to Theorem 4.12. Actually, the complete automorphism groups of A and B are solvable as well. For example, we obtain by a direct computation

$$\operatorname{Aut} A = \left\{ \begin{array}{l} \bar{x} \mapsto c_1\bar{x} + a_2\bar{x}\bar{y} + a_3\bar{y}^2, \\ \bar{y} \mapsto c_2\bar{y} + a_4\bar{x} + a_5\bar{x}\bar{y} + a_6\bar{y}^2 \end{array} \middle| \, a_i \in \mathbb{K}, c_i \in \mathbb{K}^\times \right\}.$$

Example 4.20. On the contrary, for the algebra

$$A = \mathbb{K}[x_1, \ldots, x_n]/(x_1^l, x_2^l, \ldots, x_n^l), \text{ where } l < n, \tag{4.11}$$

Schulze's criterion holds but the criterion of Theorem 4.12 does not. Note that for $n \geq 5$ the group $\operatorname{Aut} A$ is non-solvable, since it contains the subgroup of permutations of the coordinates.

We therefore see that these two criteria have distinct areas of application.

Remark 4.21. We should note that by passing to the associated graded algebra $\operatorname{gr} S$ the left-hand side of (4.1) may increase in general. Indeed, consider the ideal $\hat{I} \lhd R$ of the lowest homogeneous components of all elements in I. Then $\operatorname{gr} S = R/\hat{I}$. One can check that $\operatorname{codim}_R(I) = \operatorname{codim}_R(\hat{I})$ and $\operatorname{codim}_R(\mathfrak{m}I) \leq \operatorname{codim}_R(\mathfrak{m}\hat{I})$. This implies $\dim I/\mathfrak{m}I \leq \dim \hat{I}/\mathfrak{m}\hat{I}$. The inequality is strict, if we take for example $I = (x^2 - y^3, x^3) \lhd \mathbb{K}[\![x,y]\!]$.

4.3 Extremal algebras and Yau's theorem

Recall that by an extremal algebra we mean a finite-dimensional algebra satisfying the equality $\dim I/\mathfrak{m}I = l+n-1$ as in the setting of Theorem 4.1.

Theorem 4.22. *An extremal algebra S has a non-solvable algebra of derivations* $\operatorname{Der} S$ *if and only if it is of the form $S = S_1 \otimes S_2$, where*

$$S_1 \cong \mathbb{K}[\![x_1, x_2]\!]/(x_1^l, x_1^{l-1}x_2, \ldots, x_1x_2^{l-1}, x_2^l) \text{ for some } l \geq 2, \tag{4.12}$$
$$S_2 \cong \mathbb{K}[\![x_3, \ldots, x_n]\!]/(w_2, \ldots, w_{n-1}), \tag{4.13}$$

and $w_i \in \mathfrak{m}^l \cap \mathbb{K}[\![x_3, \ldots, x_n]\!]$ form a regular sequence.

Proof. Suppose that $S = S_1 \otimes S_2$ as above. Then the group $\mathrm{GL}(\langle x_1, x_2\rangle)$ may be embedded into $\operatorname{Aut} S_1$, hence the subalgebra S_1 carries a natural $\mathfrak{sl}_2$-representation structure. We suppose this representation is trivial on S_2.

Conversely, let the algebra of derivations $\operatorname{Der} S$ of the local algebra $S = R/I$ be non-solvable. Recall the discussion in Section 4.2. Then S is extremal if and only if the chain of inequalities (4.8) are all equalities. The first equality holds if and only if W contains exactly k simple $\mathfrak{sl}_2$-submodules, and their weights are $ln_1, \ldots, ln_k$. The second equality holds if and only if $n_1 = 1,\ n_2 = \ldots = n_k = 0$.

Under these circumstances $k = n-1$, and the simple $\mathfrak{sl}_2$-submodules of $\overline{V}$ are $\langle \bar{x}_1, \bar{x}_2\rangle, \langle \bar{x}_3\rangle, \ldots, \langle \bar{x}_n\rangle$. Then $W_{high} = \langle w_1, \ldots, w_{n-1}\rangle$, where $\mathrm{wt}(w_1) = l,\ \mathrm{wt}(w_i) = 0$ for $i = 2, \ldots, n-1$. We have

$$S = R/(w_1, f\cdot w_1, \ldots, f^l \cdot w_1, w_2, \ldots, w_{n-1}). \tag{4.14}$$

Note that the algebra $\mathfrak{sl}_2$ annihilates the series $w_2, \ldots, w_{n-1}$, hence they do not depend on x_1 and x_2 and belong to $\mathbb{K}[\![x_3, \ldots, x_n]\!] \cap \mathfrak{m}^l$. Since w_1 is the highest vector of weight l it is of the form $x_1^l g$, where $g \in \mathbb{K}[\![x_3, \ldots, x_n]\!]$. Then $f^k \cdot w_1 = x_1^{l-k}x_2^k g$.

Using the fact that $x_1^r \in I$ the equality

$$x_1^r = p_0x_1^l g + p_1 x_1^{l-1}x_2 g + \cdots + p_l x_2^l g + q_2 w_2 + \cdots + q_{n-1}w_{n-1} \tag{4.15}$$

holds for some $p_i, q_j \in R$. Letting $x_2 = x_1$, from (4.15) we obtain

$$x_1^r = (\widetilde{p}_0 + \cdots + \widetilde{p}_l)x_1^l g + \widetilde{q}_2 w_2 + \cdots + \widetilde{q}_{n-1}w_{n-1}, \tag{4.16}$$

where $\widetilde{p}_i = p_i(x_1, x_1, x_3, \ldots, x_n)$, $\widetilde{q}_j = q_j(x_1, x_1, x_3, \ldots, x_n)$. Clearly, the series $\widetilde{q}_j$ and $\widetilde{p} = \sum_{i=0}^{l} \widetilde{p}_i$ may be assumed being homogeneous in x_1. Then $\widetilde{p} = x_1^{r-l}\widehat{p}$, $\widetilde{q}_j = x_1^r \widehat{q}_j$, where $\widehat{p}, \widehat{q}_j \in \mathbb{K}[\![x_3, \ldots, x_n]\!]$. But this implies

$$x_1^l = \widehat{p}x_1^l g + x_1^l \widehat{q}_2 w_2 + \cdots + x_1^l \widehat{q}_{n-1}w_{n-1}. \tag{4.17}$$

Thus we may replace in (4.14) the $\mathfrak{sl}_2$-submodule $\langle x_1^l, x_1^{l-1}x_2, \ldots, x_1x_2^{l-1}, x_2^l\rangle$ by the $\mathfrak{sl}_2$-submodule $\langle w_1, f \cdot w_1, \ldots, f^l \cdot w_1\rangle$.

Finally, the algebra S decomposes into the tensor product $S_1 \otimes S_2$, where S_i are as required. $\square$

Now let us introduce the following well-known technical lemma for power series, see e.g. [2, Section 11.1]. For an arbitrary power series g denote by $g_{(k)}$ its kth homogeneous component.

Lemma 4.23. *Suppose $p \in \mathfrak{m}^2 \setminus \mathfrak{m}^3 \subset R$. Then up to an analytical change of coordinates we have $p = x_1^2 + \cdots + x_k^2 + q(x_{k+1}, \ldots, x_n)$, where $q \in \mathfrak{m}^3 \cap \mathbb{K}[\![x_{k+1}, \ldots, x_n]\!]$.*

Proof. The homogeneous component $p_{(2)}$ is a quadratic form, hence it is of the form $x_1^2 + \cdots + x_k^2$ up to a linear change of coordinates. Consider the following decomposition:

$$p = a_0 + a_1x_1 + a_2x_1^2 + \ldots, \tag{4.18}$$

where $a_i \in \mathbb{K}[\![x_2, \ldots, x_n]\!]$ and a_2 is invertible.

Now consider a change of coordinates $\varphi : x_1 \mapsto x_1 + g$, $x_2 \mapsto x_2$, $\ldots$, $x_n \mapsto x_n$, where $g \in \mathfrak{m} \cap \mathbb{K}[\![x_2, \ldots, x_n]\!]$. Thus,

$$\varphi(p) = \widetilde{a}_0 + \widetilde{a}_1x_1 + \widetilde{a}_2x_1^2 + \ldots \tag{4.19}$$

for some $\widetilde{a}_i \in \mathbb{K}[\![x_2, \ldots, x_n]\!]$. In this case

$$\widetilde{a}_1 = a_1 + 2ga_2 + 3g^2a_3 + \ldots = a_2\left(\frac{a_1}{a_2} + 2g + 3g^2\frac{a_3}{a_2} + \ldots\right). \tag{4.20}$$

Since $a_1 \in \mathfrak{m}$, we can choose g such that $\widetilde{a}_1 = 0$. Indeed, let us break up the series in parentheses on the right hand side of (4.20) into homogeneous components of the form $2g_{(k)} + P_k$, where P_k depends only on the first $k-1$ homogeneous components of g. Thus, by induction, all the homogeneous components of the series g are uniquely determined. Note that $\widetilde{a}_2 = a_2 + 3a_3g + 6a_4g^2 + \ldots$ is still invertible.

Therefore we may suppose that $a_1 = 0$. Consider a substitution $x_1 \mapsto x_1b$ where $b^2 = a_2^{-1}$ (other coordinates are untouched). It is easily seen that the required series b exists and is actually a change of coordinates, since b is invertible. Thus we may suppose as well that $a_2 = 1$.

Finally, consider a change of coordinates $x_1 \mapsto x_1 + b_2x_1^2 + b_3x_1^3 + \ldots$. Similarly to the choice of g, we may find b_i such that $p \mapsto a_0 + x_1^2$. By induction on n we may assume that $a_0(x_2, \ldots, x_n)$ is of the required form. Then p is of the required form too. $\square$

Proof of Theorem 4.5. According to Yau's remark in [24] we may assume $p \in \mathfrak{m}^3$. Indeed, let $p \in \mathfrak{m}^2 \setminus \mathfrak{m}^3$. By Lemma 4.23 we have $p = x_1^2 + \ldots + x_k^2 + q(x_{k+1}, \ldots, x_n)$ up to the a change of coordinates. Since $\frac{\partial p}{\partial x_i} = 2x_i$ for $i = 1, \ldots, k$, the equality $\mathbb{K}[\![x_1, \ldots, x_n]\!]/(p, J(p)) \cong \mathbb{K}[\![x_{k+1}, \ldots, x_n]\!]/(q, J(q))$ holds, and we may replace p by a series $q \in \mathfrak{m}^3 \cap \mathbb{K}[\![x_{k+1}, \ldots, x_n]\!]$.

If $p \in \mathfrak{m}^3$ then $I = (p, J(p)) \subset \mathfrak{m}^2$ and $l \geq 2$. We must show that the Yau algebra $\operatorname{Der} S$ of the moduli algebra $S = \mathbb{K}[\![x_1, \ldots, x_n]\!]/I$ is solvable. Suppose the contrary.

Note that $\dim(I/\mathfrak{m}I) \leq n + 1 \leq n + l - 1$. Hence the moduli algebra either satisfies the inequality of Schulze's criterion and $\operatorname{Der} S$ is solvable, or it is extremal and $l = 2$. In the latter case we may apply Theorem 4.22 and so $S = S_1 \otimes S_2$, where $S_1 = \mathbb{K}[\![x_1, x_2]\!]/(x_1^2, x_1x_2, x_2^2)$ and $S_2 = \mathbb{K}[\![x_3, \ldots, x_n]\!]/(w_2, \ldots, w_{n-1})$.

It is easily seen that the homogeneous component $p_{(3)}$ has a form $p_1(x_1, x_2) + p_2(x_3, \ldots, x_n)$. Indeed, otherwise $J(p)$ would contain a series with a term of the form x_ix_j, where $i \in \{1, 2\}, j \in \{3, \ldots, n\}$, and hence

$$\langle x_1^2, x_1x_2, x_2^2\rangle \subset (p, J(p)) \cap (\mathbb{K}[x_1, x_2])_2 \subset \left\langle \frac{\partial p_1}{\partial x_1}, \frac{\partial p_1}{\partial x_2} \right\rangle, \tag{4.21}$$

a contradiction. $\square$

4.4 The global case and Halperin's conjecture

Let $S = \mathbb{K}[x_1, \ldots, x_n]/I$ be a finite-dimensional algebra, not necessarily local. Suppose that it contains s maximal ideals $\bar{\mathfrak{m}}_1, \ldots, \bar{\mathfrak{m}}_s$. Then there exists an integer $k \in \mathbb{N}$ such that $\prod_{i=1}^s \bar{\mathfrak{m}}_i^k = 0$. Since the ideals $\bar{\mathfrak{m}}_i^k$ are coprime we have $\bigcap_{i=1}^s \bar{\mathfrak{m}}_i^k = \prod_{i=1}^s \bar{\mathfrak{m}}_i^k$. Finally, by [5, Theorem 8.7] the algebra S can be decomposed into a direct product of local subalgebras as follows,

$$S \cong \prod_{i=1}^{s} S/\bar{\mathfrak{m}}_i^k. \tag{4.22}$$

In addition, any decomposition of S into the local subalgebras is of the form (4.22), i.e. they are uniquely determined up to isomorphism.

On the other hand, there is a unique maximal decomposition of the identity

$$1 = e_1 + \ldots + e_t \tag{4.23}$$

into a sum of orthogonal idempotents, i.e. $e_ie_i = e_i$ for all i and $e_ie_j = 0$ for $i \neq j$, see e.g. [6, Section II.5]. Then we have the decomposition

$$S = \bigoplus_{i=1}^{t} e_iS, \tag{4.24}$$

where the subalgebras e_iS are indecomposable and hence local. This means that

$$S_i = e_iS \cong S/\bar{\mathfrak{m}}_i^k \tag{4.25}$$

up to permutation of indexes, and $t = s$. Thus, we have a uniquely determined decomposition (4.24) of S into local subalgebras.

Proposition 4.24. $\operatorname{Aut}^\circ S = \operatorname{Aut}^\circ S_1 \times \ldots \times \operatorname{Aut}^\circ S_s$.

Proof. Since the decomposition (4.23) is unique, the primary idempotents e_i in this decomposition are preserved by $\operatorname{Aut}^\circ S$ as well as the subalgebras S_i. Since $S_iS_j = 0$ for $i \neq j$, the required statement holds. $\square$

Denote by $\mathfrak{m}_i = (x_1 - a_{1i}, \ldots, x_n - a_{ni})$, $i = 1 \ldots s$, the maximal ideals in $\mathbb{K}[x_1, \ldots, x_n]$ corresponding to $\bar{\mathfrak{m}}_1, \ldots, \bar{\mathfrak{m}}_s \lhd S$. Then $\bigcap_{i=1}^s \mathfrak{m}_i^k \subset I$. Let us introduce the corresponding algebras of formal power series $R_i = \mathbb{K}[\![x_1 - a_{1i}, \ldots, x_n - a_{ni}]\!]$. The ideal $(\mathfrak{m}_j) \lhd R_i$ coincides with R_i unless $i = j$. In the latter case $(\bar{\mathfrak{m}}_i) \lhd R_i$ is the maximal ideal which we denote by $\widetilde{\mathfrak{m}}_i$. Therefore the inclusion $\widetilde{\mathfrak{m}}_i^k = \bigcap_{i=1}^s \left(\mathfrak{m}_i^k\right) \subset (I) \lhd R_i$ holds, and

$$S_i \cong S/\bar{\mathfrak{m}}_i^k \cong \mathbb{K}[x_1, \ldots, x_n]/(I, \mathfrak{m}_i^k) \cong R_i/(I). \tag{4.26}$$

Taking into account Proposition 4.24, we obtain the following solvability criterion.

Theorem 4.25. *The identity component* $\operatorname{Aut}^\circ S$ *of the finite-dimensional algebra* $S = \mathbb{K}[x_1, \ldots, x_n]/I$ *with maximal ideals* $\bar{\mathfrak{m}}_1, \ldots, \bar{\mathfrak{m}}_s$ *is solvable if and only if the identity component* $\operatorname{Aut}^\circ S_i$ *of the local algebra* $S_i = R_i/(I)$ *is solvable for any* $i = 1, \ldots, s$.

We can now prove Halperin's conjecture.

Proof of Conjecture 4.2. Suppose $S = \mathbb{K}[x_1, \ldots, x_n]/(f_1, \ldots, f_n)$. Since the local subalgebra $S_i = R_i/(f_1, \ldots, f_n)$ is a local complete intersection, the group $\operatorname{Aut}^\circ S_i$ is solvable by Corollary 4.4 for each i. Then by Theorem 4.25 the group $\operatorname{Aut}^\circ S$ is solvable as well. $\square$

Theorem 4.25 allows us to deduce the following globalization of Schulze's criterion.

Corollary 4.26. *Suppose we have an ideal* $I \subset \mathbb{K}[x_1, \ldots, x_n]$ *with* m *generators and an integer* $l > 1$ *such that the following holds:*

- *The quotient algebra* $S = \mathbb{K}[x_1, \ldots, x_n]/I$ *is finite-dimensional,*
- *For any maximal ideal* $\mathfrak{m} \subset \mathbb{K}[x_1, \ldots, x_n]$ *there holds either* $I \not\subset \mathfrak{m}$ *or* $I \subset \mathfrak{m}^l$*,*

- *The inequality $m < n + l - 1$ holds.*

Then the identity component $\operatorname{Aut}^\circ S$ *is solvable.*

Proof. Let as before $\mathfrak{m}_1, \dots, \mathfrak{m}_s \lhd \mathbb{K}[x_1, \dots, x_n]$ be the only ideals containing I. By Corollary 4.14, $\dim(I/\mathfrak{m}_i I) \leq m < n+l-1$ and Schulze's criterion is applicable for the algebras $R_i/(I)$, $i = 1, \dots, s$. Therefore $\operatorname{Aut}^\circ S$ is solvable by Theorem 4.25. □

4.5 Automorphism subgroups and dimension bounds

As usual we assume the ideal $I \lhd R$ contains $\mathfrak{m}^l$, where $l \geq 2$, and the algebra $S = R/I$ is finite-dimensional and local with the maximal ideal $\bar{\mathfrak{m}} = (\bar{x}_1, \dots, \bar{x}_n)$.

Recall that the sum of all minimal ideals of a finite-dimensional algebra S is called the *socle* $\operatorname{Soc} S$. It is invariant under endomorphisms of S. The *annihilator* of an arbitrary subset $X \subset S$ is the ideal $\operatorname{Ann} X = \{z \in S \mid zX = 0\}$.

Lemma 4.27. $\operatorname{Soc} S = \operatorname{Ann} \bar{\mathfrak{m}}$.

Proof. Consider an arbitrary minimal ideal $J \subset \operatorname{Soc} S$. Obviously, $\bar{\mathfrak{m}} J \subset J$. But $\bar{\mathfrak{m}} J \neq J$ by Nakayama's Lemma. Thus, $\bar{\mathfrak{m}} J = 0$ and $\operatorname{Soc} S \subset \operatorname{Ann} \bar{\mathfrak{m}}$.

Suppose $z \in \operatorname{Ann} \bar{\mathfrak{m}}$. Then $zS = \{z(c + w) \mid c \in \mathbb{K}, w \in \bar{\mathfrak{m}}\} = \{cz \mid c \in \mathbb{K}\}$, so the principal ideal (z) is one-dimensional and minimal. This implies $\operatorname{Ann} \bar{\mathfrak{m}} \subset \operatorname{Soc} S$. □

Assuming that S is graded, Y.-J. Xu and S. S.-T. Yau found a bound for the dimension of the group $\operatorname{Aut} S$ as follows (see [22, Proposition 2.3]):

$$\dim \operatorname{Aut} S \geq \dim S - \dim \operatorname{Soc} S. \tag{4.27}$$

In Theorem 4.30 we introduce a lower bound without this assumption.

Definition 4.28. Let us call the *lower socle* of the algebra S the ideal $\operatorname{LSoc} S = \operatorname{Soc} S \cap \bar{\mathfrak{m}}^2$. We may choose a subspace $\operatorname{USoc} S \subset \operatorname{Soc} S$ such that

$$\operatorname{Soc} S = \operatorname{USoc} S \oplus \operatorname{LSoc} S, \tag{4.28}$$

and call it an *upper socle*. Note that the choice of the upper socle is not canonical. However, up to a change of coordinates we may suppose $\operatorname{USoc} S \subset \langle \bar{x}_1, \dots, \bar{x}_n \rangle$.

Proposition 4.29. *The automorphism group of a finite-dimensional local algebra S contains a unipotent subgroup $U \subset \operatorname{Aut} S$ with*

$$\dim U = \dim(\operatorname{LSoc} S)\cdot\dim(\bar{\mathfrak{m}}/\bar{\mathfrak{m}}^2)+\dim(\operatorname{USoc} S)\cdot(\dim(\bar{\mathfrak{m}}/\bar{\mathfrak{m}}^2)-\dim(\operatorname{USoc} S)). \tag{4.29}$$

Proof. Suppose that $\operatorname{USoc} S = \langle \bar{x}_1, \ldots, \bar{x}_s \rangle$. Consider the unipotent subgroup of linear transformations

$$U = \{u \colon \bar{x}_1 \mapsto \bar{x}_1 + F_1, \ldots, \bar{x}_n \mapsto \bar{x}_n + F_n \mid \\ F_1, \ldots, F_s \in \operatorname{LSoc} S,\ F_{s+1}, \ldots, F_n \in \operatorname{Soc} S\} \subset \mathrm{GL}(S), \quad (4.30)$$

acting trivially on the subspace $\langle 1 \rangle \oplus \bar{\mathfrak{m}}^2$. It is easily seen that

$$u(\bar{x}_i)u(\bar{x}_j) = (\bar{x}_i + F_i)(\bar{x}_j + F_j) = \bar{x}_i\bar{x}_j = u(\bar{x}_i\bar{x}_j) \text{ for } i, j \in \{1, \ldots, n\}, u \in U. \quad (4.31)$$

Hence

$$u(a)u(b) = ab = u(ab) \text{ for } a, b \in \bar{\mathfrak{m}}, u \in U. \quad (4.32)$$

Therefore the U-action is consistent with the multiplication in S, so $U \subset \operatorname{Aut} S$. Finally,

$$\dim U = s \cdot \dim(\operatorname{LSoc} S) + (n - s) \cdot \dim(\operatorname{Soc} S) = \\ n \cdot \dim(\operatorname{LSoc} S) + (n - s) \cdot \dim(\operatorname{USoc} S). \quad (4.33)$$

□

Theorem 4.30. $\dim \operatorname{Aut} S \geq \dim(\bar{\mathfrak{m}}/\bar{\mathfrak{m}}^2) \cdot \dim \operatorname{Soc} S$.

Proof. Consider the subgroup $G = \mathrm{GL}(\operatorname{USoc} S) \subset \operatorname{Aut} S$. Along with the subgroup U from Proposition 4.29 it generates a subgroup $GU \subset \operatorname{Aut} S$. To prove the inequality

$$\dim GU \geq \dim G + \dim U \quad (4.34)$$

it suffices to look at the tangent algebras of G and U. Indeed, it is easy to see that they have zero intersection, and hence

$$\dim GU = \dim(\operatorname{Lie} GU) \geq \dim(\operatorname{Lie} G) + \dim(\operatorname{Lie} U) = \\ \dim G + \dim U = (\dim(\operatorname{USoc} S))^2 + \dim(\operatorname{LSoc} S) \cdot \dim(\bar{\mathfrak{m}}/\bar{\mathfrak{m}}^2) + \\ \dim(\operatorname{USoc} S) \cdot (\dim(\bar{\mathfrak{m}}/\bar{\mathfrak{m}}^2) - \dim(\operatorname{USoc} S)) = \dim(\operatorname{Soc} S) \cdot \dim(\bar{\mathfrak{m}}/\bar{\mathfrak{m}}^2). \quad (4.35)$$

□

Corollary 4.31. *The group* $\operatorname{Aut} S$ *is infinite if* $S \neq \mathbb{K}$.

Clearly, the automorphism group almost always contains a rather big unipotent subgroup. A natural question arises if the whole automorphism group may be unipotent. Consider the following example.

Example 4.32. We claim that the group $\operatorname{Aut} S$ of the following local algebra

$$S = \mathbb{K}[x,y]/I, \quad I = (y^5, (x+y)^6, x^5 - x^3y^3, x^4y), \tag{4.36}$$

is unipotent. Indeed, the Gröbner basis of the ideal I with respect to the homogeneous lexicographic order with $x \prec y$ is

$$\{x^6, y^5, x^3y^3 - x^5, 3x^2y^4 + 4x^5, x^4y\}. \tag{4.37}$$

Clearly, $\mathfrak{m}^7 \subset I$. Let $\bar{x}, \bar{y}$ be the images of x, y respectively under factorization by I. The basis of the algebra S is as follows:

$$\begin{array}{llllll}
1 & \bar{x} & \bar{x}^2 & \bar{x}^3 & \bar{x}^4 & (\bar{x}^5) \\
\bar{y} & \bar{x}\bar{y} & \bar{x}^2\bar{y} & \bar{x}^3\bar{y} & & \\
\bar{y}^2 & \bar{x}\bar{y}^2 & \bar{x}^2\bar{y}^2 & \bar{x}^3\bar{y}^2 & & \\
\bar{y}^3 & \bar{x}\bar{y}^3 & \bar{x}^2\bar{y}^3 & (\bar{x}^3\bar{y}^3) & & \\
\bar{y}^4 & \bar{x}\bar{y}^4 & (\bar{x}^2\bar{y}^4), & & &
\end{array}$$

where $-\frac{3}{4}\bar{x}^2\bar{y}^4 = \bar{x}^3\bar{y}^3 = \bar{x}^5$.

Consider an arbitrary automorphism $\varphi \in \operatorname{Aut} S$, with

$$\varphi(\bar{x}) = a_{11}\bar{x} + a_{12}\bar{y} + h_1(\bar{x}, \bar{y}), \quad h_1 \in \mathfrak{m}^2, \tag{4.38}$$
$$\varphi(\bar{y}) = a_{21}\bar{x} + a_{22}\bar{y} + h_2(\bar{x}, \bar{y}), \quad h_2 \in \mathfrak{m}^2. \tag{4.39}$$

Note that $\bar{y}$ is the only linear polynomial whose 5th power is zero. Then $\varphi(\bar{y}^5) = 0$ implies $a_{21} = 0$. On the other hand, $\bar{x}$ is the only linear polynomial whose 5th degree is non-zero and belongs to $\bar{\mathfrak{m}}^6$. Therefore,

$$\varphi(\bar{x}) = a_{11}\bar{x} + h_1(\bar{x}, \bar{y}), \quad h_1 \in \mathfrak{m}^2, \tag{4.40}$$
$$\varphi(\bar{y}) = a_{22}\bar{y} + h_2(\bar{x}, \bar{y}), \quad h_2 \in \mathfrak{m}^2, \tag{4.41}$$

where $a_{11}, a_{22} \neq 0$ as φ is invertible. Notice that $\varphi((\bar{x}+\bar{y})^6) = 0$ if and only if $a_{11} = a_{22} = c$. Finally, $\varphi(x^3y^3 - x^5) = c^6x^3y^3 - c^5x^5 = 0$. This implies that $c = 1$.

Hence for an arbitrary element $z \in \bar{\mathfrak{m}}^i$ the inclusion $(\operatorname{Aut} S) \cdot z \subset z + \bar{\mathfrak{m}}^{i+1}$ holds. Thus, the group $\operatorname{Aut} S$ is unipotent.

Part II

Infinite transitivity and flexibility of affine varieties

Chapter 5

General facts

5.1 Introduction

In this part we work over an algebraically closed field of characteristic zero, except for Section 5.2, where the characteristic might be arbitrary. Recall some useful notions introduced in [29] and [30].

An action of a group G on a set A is said to be *m-transitive* if for every two tuples of pairwise distinct points $(a_1, \ldots, a_m)$ and $(a'_1, \ldots, a'_m)$ in A there exists $g \in G$ such that $g \cdot a_i = a'_i$ for $i = 1, \ldots, m$. An action which is m-transitive for all $m \in \mathbb{Z}_{>0}$ is called *infinitely transitive*.

Let Y be an algebraic variety of dimension ≥ 2. Consider a regular action $\mathbb{G}_a \times Y \to Y$ of the additive group $\mathbb{G}_a = (\mathbb{K}, +)$ of the ground field on Y. The image, say, L of $\mathbb{G}_a$ in the automorphism group $\operatorname{Aut} Y$ is a one-parameter unipotent subgroup. We let $\operatorname{SAut}(Y)$ denote the subgroup of $\operatorname{Aut}(Y)$ generated by all its one-parameter unipotent subgroups. It is called a *special automorphism group*. Evidently, $\operatorname{SAut}(Y)$ is a normal subgroup of $\operatorname{Aut}(Y)$.

An affine algebraic variety X is called *flexible* if the tangent space of X at any smooth point is spanned by the tangent vectors to the orbits of one-parameter unipotent group actions. This notion was introduced in [30] and further developed in [29]. The following theorem explains the significance of the flexibility concept.

Theorem 5.1 ([29, Theorem 0.1])**.** *Let X be an affine algebraic variety of dimension ≥ 2. Then the following conditions are equivalent.*

1. *The variety X is flexible;*
2. *the group $\operatorname{SAut} X$ acts transitively on the smooth locus X_{reg} of X;*
3. *the group $\operatorname{SAut} X$ acts infinitely transitively on X_{reg}.*

The first examples of flexible varieties are described in [30], namely the following three classes: affine cones over flag varieties, non-degenerate toric varieties of dimension ≥ 2, and suspensions over flexible varieties.

Definition 5.2. A derivation D on the algebra of regular functions $\mathbb{K}[Y] = \Gamma(Y, \mathcal{O}_Y)$ of an algebraic variety Y is called *locally nilpotent*, if for any $f \in \mathbb{K}[Y]$ there exists $n \in \mathbb{N}$ such that $D^n(f) = 0$.

For an affine variety Y over a field of characteristic zero there exists a canonical one-to-one correspondence between locally nilpotent derivations (or LNDs, for short) on $\mathbb{K}[Y]$ and $\mathbb{G}_a$-actions on Y. Indeed, a regular action $\mathbb{G}_a \times Y \to Y$ defines a structure of a rational $\mathbb{G}_a$-algebra on $\mathbb{K}[Y]$, and the infinitesimal generator of this action is a locally nilpotent derivation D on $\mathbb{K}[Y]$. Conversely, for a locally nilpotent derivation D on $\mathbb{K}[Y]$ the one-parameter group $\{\exp(tD) \mid t \in \mathbb{K}\}$ is an algebraic subgroup of $\operatorname{Aut} Y$, see [43].

If we allow Y to be quasiaffine or the ground field to be of positive characteristic, then this correspondence fails to exist. For a quasiaffine variety Y any $\mathbb{G}_a$-action corresponds to an LND, but not every LND induces a $\mathbb{G}_a$-action. To see this, we may take $Y = \mathbb{A}^2 \setminus \{0\}$ with the algebra of regular functions $\mathbb{K}[Y] = \mathbb{K}[x, y]$ and a derivation $D = \frac{\partial}{\partial x}$. The $\mathbb{G}_a$-action $\{\exp(tD)\} = \{x \mapsto x + t, y \mapsto y\}$ on $\mathbb{K}[Y]$ does not preserve the maximal ideal of the origin.

If the field is of characteristic $p > 0$, the situation is even worse. For example, if we consider a non-trivial $\mathbb{G}_a$-action $\{x \mapsto x + t^p, y \mapsto y \mid t \in \mathbb{K}\}$ on $Y = \mathbb{A}^2$, then the corresponding LND is zero. Moreover, the exponential map from LNDs to $\mathbb{G}_a$-actions does not exist. Finally, it is not known whether the equivalence of notions of flexibility and transitivity as in Theorem 5.1 holds, but the following example provides a hint on a possible obstruction.

Example 5.3. Consider the following two $\mathbb{G}_a$-actions on the affine plane $Y = \mathbb{A}^2$:

$$h_1 \colon \mathbb{G}_a \times Y \to Y,\ (t, x, y) \mapsto (x + t, y); \tag{5.1}$$

$$h_2 \colon \mathbb{G}_a \times Y \to Y,\ (t, x, y) \mapsto (x + t, y + t^p). \tag{5.2}$$

They commute, so the conjugations are trivial. They are transitive on Y, yet at each point $q \in Y$ the tangent vectors of their orbits coincide and span only a one-dimensional subspace. The same holds if we take all replicas of h_1 and h_2 and obtain a saturated subgroup of $\operatorname{Aut} Y$, see Section 5.2 for the definitions.

Nevertheless, the equivalence of transitivity and infinite transitivity does hold for quasiaffine varieties over a closed field of arbitrary characteristic. The proof is provided in Section 5.2. It is based on the original proof of Theorem 5.1 and a proof for quasiaffine varieties in charactestic zero in [31]. We should also mention [41, Theorem 1.11], which is a generalization of Theorem 5.1 for quasiaffine varieties.

Let us introduce some facts about the kernel of an LND.

Lemma 5.4 ([57, Lemma 1.7]). *Let A be a finitely generated normal domain over a field of characteristic 0. If D and D' are two LNDs on A, then the following hold:*

(i) $\operatorname{Ker} D$ *is a normal subdomain of codimension one.*

(ii) $\operatorname{Ker} D$ *is factorially closed i.e.,* $ab \in \operatorname{Ker} D \setminus \{0\} \Rightarrow a, b \in \operatorname{Ker} D$.

(iii) If $a \in A$ *is invertible, then* $a \in \operatorname{Ker} D$.

(iv) If $\operatorname{Ker} D = \operatorname{Ker} D'$, *then there exist* $f, f' \in \operatorname{Ker} D$ *such that* $f'D = fD'$.

(v) For $a \in A$, $Da \in (a) \Rightarrow Da = 0$.

(vi) If $a \in \operatorname{Ker} D$, *then* aD *is again a LND.*

If D is an LND assigned to a $\mathbb{G}_a$-action H on a quasiaffine variety Y and $f \in \operatorname{Ker} D$, then the derivation fD is locally nilpotent and it corresponds to a $\mathbb{G}_a$-action on Y with the same orbits on $Y \setminus \operatorname{div}(f)$. This new action fixes each point of the divisor $\operatorname{div}(f)$. The one-parameter subgroup of $\operatorname{SAut}(Y)$ defined by fD is called a *replica* of the subgroup generated by D. Note that we can define replicas without using LNDs. Indeed, let H be defined by a morphism

$$h : \mathbb{G}_a \times Y \to Y,\ (t, y) \mapsto h(t, y).$$

Define the replica fH w.r.t. an H-invariant function f by the morphism

$$h_f : \mathbb{G}_a \times Y \to Y,\ (t, y) \mapsto h(f(y)t, y).$$

This definition is concurrent with the previous in characteristic zero and preserves the same conditions on orbits in positive characteristic.

We should note that the infinite transitivity of $\operatorname{SAut} Y$ on Y_{reg} implies that $\operatorname{SAut} Y$ is of infinite dimension (and so, is not contained in any algebraic subset). This can be easily seen if we consider the family of all replicas of a $\mathbb{G}_a$-action on Y corresponding to an LND D. Indeed, since $\operatorname{Ker} D \subset \mathbb{K}[Y]$ is of codimension one and $\dim Y \geq 2$, this family induces an infinite-dimensional subgroup in $\operatorname{Aut} Y$. The other trivial observation is that the one-dimensional orbits of replicas coincide, hence they are determined by $\operatorname{Ker} D$. The elements of the kernel are exactly the invariant functions for any $\mathbb{G}_a$-action in the considered family.

It is often of great interest to know if the group $\operatorname{SAut} Y$ acts with an open orbit on a variety Y, which is a weaker condition than in Theorem 5.1. In this case the following invariant has proved to be useful. The subfield of rational functions annihilated by all locally nilpotent derivations

$$\operatorname{FML}(Y) = \bigcap_{\delta \in \operatorname{LND}(Y)} \operatorname{Quot}(\ker \delta) \subset \mathbb{K}(Y)$$

is called a *field Makar–Limanov invariant.* The invariant is said to be *trivial* if it equals $\mathbb{K}$.

Proposition 5.5 ([29, Proposition 5.1])**.** *The field Makar–Limanov invariant* $\mathrm{FML}(Y)$ *is trivial if and only if the group* $\mathrm{SAut}\,Y$ *acts on* Y *with an open orbit.*

5.2 Transitivity and infinite transitivity of special automorphisms

In this section we allow the algebraically closed field $\mathbb{K}$ to be of arbitrary characteristic. We say that a subgroup G of $\mathrm{Aut}(Y)$ is *algebraically generated* if it is generated as an abstract group by a family $\mathcal{G}$ of connected algebraic[1] subgroups of $\mathrm{Aut}(Y)$.

Proposition 5.6. *[29, Proposition 1.5] Let* $G \subset \mathrm{Aut}\,Y$ *be a subgroup algebraically generated by a family* $\mathcal{G}$ *of connected algebraic subgroups acting effectively on* Y*. Then there are (not necessarily distinct) subgroups* $H_1, \ldots, H_s \in \mathcal{G}$ *such that*

$$G.x = (H_1 \cdot H_2 \cdot \ldots \cdot H_s) \cdot x \quad \forall x \in X. \tag{5.3}$$

A sequence $\mathcal{H} = (H_1, \ldots, H_s)$ satisfying condition (5.3) of Proposition 5.6 is called *complete.*

Let us say that a subgroup $G \subset \mathrm{SAut}(Y)$ is *saturated* if it is generated by one-parameter unipotent subgroups and there is a complete sequence $(H_1, \ldots, H_s)$ of one-parameter unipotent subgroups in G such that G contains all replicas of $H_1, \ldots, H_s$. In particular, $G = \mathrm{SAut}(X)$ is a saturated subgroup.

Theorem 5.7. *Let* Y *be an irreducible quasiaffine algebraic variety of dimension* ≥ 2 *and let* $G \subset \mathrm{SAut}(Y)$ *be a saturated subgroup, which acts with an open orbit* $O \subset Y$*. Then* G *acts on* O *infinitely transitively.*

Corollary 5.8. *Let* Y *be an irreducible quasiaffine algebraic variety of dimension* ≥ 2*. Then the action of* $\mathrm{SAut}\,Y$ *on* Y_{reg} *is transitive if and only if it is infinitely transitive.*

Remark 5.9. Let H be a one-parameter unipotent subgroup of G. According to [62, Theorem 3.3], the field of rational invariants $\mathbb{K}(Y)^H$ is the field of fractions of the algebra $\mathbb{K}[Y]^H$ of regular invariants. Hence, by Rosenlicht's Theorem (see [62, Proposition 3.4]), regular invariants separate orbits on an H-invariant open dense subset $U(H)$ in Y. Furthermore, $U(H)$ can be chosen to be contained in O and consisting of 1-dimensional H-orbits.

[1] not necessarily affine.

For the remaining part of this section we fix the following notation. Let $H_1, \dots, H_s$ be a complete sequence of one-parameter unipotent subgroups in G. We choose subsets $U(H_1), \dots, U(H_s) \subset O$ as in Remark 5.9 and let

$$V = \bigcap_{k=1}^{s} U(H_k)\,.$$

In particular, V is open and dense in O. We say that a set of points $x_1, \dots, x_m$ in Y is *regular*, if $x_1, \dots, x_m \in V$ and $H_k \cdot x_i \neq H_k \cdot x_j$ for all $i, j = 1, \dots, m$, $i \neq j$, and all $k = 1, \dots, s$.

Remark 5.10. For any H_k, any 1-dimensional H_k-orbits $O_1, \dots, O_r$ intersecting V and any $p = 1, \dots, s$ we may choose a replica $H_{k,p}$ such that all O_q but O_p are pointwise $H_{k,p}$-fixed. To this end, we find H_k-invariant functions $f_{k,p,p'}$ such that $f_{k,p,p'}|_{O_p} = 1$, $f_{k,p,p'}|_{O_{p'}} = 0$. Then we take the replicas

$$H_{k,p} = (\prod_{p' \neq p} f_{k,p,p'})H_k.$$

Lemma 5.11. *For every subset $x_1, \dots, x_m \in O$ there exists an element $g \in G$ such that the set $g \cdot x_1, \dots, g \cdot x_m$ is regular.*

Proof. For any x_i there holds $V \subset O = H_1 \cdots H_s \cdot x_i$. The condition $h_1 \cdots h_s \cdot x \in V$ is open and nonempty, hence we obtain an open subset $W \subset H_1 \times \ldots \times H_s$ such that $h_1 \cdots h_s \cdot x_i \in V$ for any $(h_1, \dots, h_s) \in W$ and any x_i.

So we may suppose that $x_1, \dots, x_m \in V$. Let N be the number of triples (i, j, k) such that $i \neq j$ and $H_k \cdot x_i = H_k \cdot x_j$. If $N = 0$, then the lemma is proved. Assume that $N \geq 1$ and fix such a triple (i, j, k).

There exists l such that $H_k \cdot x_i$ has at most finite intersection with H_l-orbits; otherwise $H_k \cdot x_i$ is invariant with respect to all $H_1, \dots, H_s$, a contradiction with the condition $\dim O \geq 2$.

We claim that there is a one-parameter subgroup H in G such that

$$H_k \cdot (h \cdot x_i) \neq H_k \cdot (h \cdot x_j) \quad \text{for all but finitely many elements } h \in H. \tag{5.4}$$

Since this condition is determined by a finite set of H_k-invariant functions, either it holds or $H_k \cdot (h \cdot x_i) = H_k \cdot (h \cdot x_j)$ for all $h \in H$.

Assume that $H_l \cdot x_i \neq H_l \cdot x_j$. By Remark 5.10 there exists a replica H_l' such that $H_l' \cdot x_i = x_i$, but $H_l' \cdot x_j = H_l \cdot x_j$. We take $H = H_l'$, and condition (5.4) is fulfilled.

Assume now the contrary. Let $\psi_l \colon \mathbb{K} \cong H_l$ be the homomorphism of additive groups. Then there exists $c \in \mathbb{K}$ such that $\psi_l(c) \cdot x_i = x_j$. There also exists an H_k-invariant function $f \in \mathbb{K}[Y]$, which restriction to $H_l \cdot x_i$ is a non-constant polynomial $p(t) = f(\psi_l(t) \cdot x_i)$. If $H_k \cdot (\psi_l(t) \cdot x_i) = H_k \cdot (\psi_l(t) \cdot x_j)$ for a general $t \in \mathbb{K}$, then

$$p(t) = f(\psi_l(t) \cdot x_i) = f(\psi_l(t) \cdot x_j) = f(\psi_l(t+a) \cdot x_i) = p(t+a) \qquad (5.5)$$

for a general $t \in \mathbb{K}$, which is impossible. Therefore, we may take $H = H_l$ in order to fulfill condition (5.4).

Finally, the following conditions are open and nonempty on H:

(C1) $h \cdot x_1, \dots, h \cdot x_m \in V$;

(C2) if $H_p \cdot x_{i'} \neq H_p \cdot x_{j'}$ for some p and $i' \neq j'$, then $H_p \cdot (h \cdot x_{i'}) \neq H_p \cdot (h \cdot x_{j'})$.

Hence there exists $h \in H$ satisfying (C1), (C2), and condition (5.4). We conclude that for the set $(h \cdot x_1, \dots, h \cdot x_m)$ the value of N is smaller, and proceed by induction. □

Lemma 5.12. *Let $x_1, \dots, x_m$ be a regular set and $G(x_1, \dots, x_{m-1})$ be the intersection of the stabilizers of the points $x_1, \dots, x_{m-1}$ in G. Then the orbit $G(x_1, \dots, x_{m-1}) \cdot x_m$ contains an open subset in O.*

Proof. We claim that there is a nonempty open subset $U \subset H_1 \times \dots \times H_s$ such that for every $(h_1, \dots, h_s) \in U$ we have

$$h_1 \dots h_s \cdot x_m = g \cdot x_m \quad \text{for some } g \in G(x_1, \dots, x_{m-1}).$$

Indeed, let Z be the union of orbits $H_k \cdot x_i$, $k = 1, \dots, s$, $i = 1, \dots, m-1$. The set $V \setminus Z$ is open and contains x_m. Let U be the set of all $(h_1, \dots, h_s)$ such that $h_r \dots h_s \cdot x_m \in V \setminus Z$ for any $r = 1, \dots, s$. Then U is open and nonempty. Let us show that for any $(h_1, \dots, h_s) \in U$ and any $r = 1, \dots, s$ the point $h_r \dots h_s \cdot x_m$ is in the orbit $G(x_1, \dots, x_{m-1}) \cdot x_m$. Assume that $h_{r+1} \dots h_s \cdot x_m \in G(x_1, \dots, x_{m-1}) \cdot x_m$. By Remark 5.10, there is a replica H_r' of the subgroup H_r which fixes $x_1, \dots, x_{m-1}$ and such that the orbits

$$H_r \cdot (h_{r+1} \dots h_s \cdot x_m) \quad \text{and} \quad H_r' \cdot (h_{r+1} \dots h_s \cdot x_m)$$

coincide. Then H_r' is contained in $G(x_1, \dots, x_{m-1})$ and the point $h_r h_{r+1} \dots h_s \cdot x_m$ is in the orbit $G(x_1, \dots, x_{m-1}) \cdot x_m$ for any $h_r \in H_r$. The claim is proved.

Now the image of the dominant morphism

$$U \to O, \quad (h_1, \dots, h_s) \mapsto h_1 \dots h_s \cdot x_m$$

contains an open subset in O. □

Proof of Theorem 5.7. Let $(x_1, \dots, x_m)$ and $(y_1, \dots, y_m)$ be two sets of pairwise distinct points in O. We have to show that there is an element $g \in G$ such that $g \cdot x_1 = y_1, \dots, g \cdot x_m = y_m$.

We argue by induction on m. If $m = 1$, then the claim is obvious. If $m > 1$, then by inductive hypothesis there exists $g' \in G$ such that $g' \cdot x_1 = y_1, \dots, g' \cdot x_{m-1} = y_{m-1}$. If $g' \cdot x_m = y_m$, the assertion is proved. Assume that $g' \cdot x_m \neq y_m$. By Lemma 5.11, there exists $g'' \in G$ such that the set

$$g'' \cdot y_1,\ \dots, g'' \cdot y_{m-1},\ g'' \cdot y_m,\ g''g' \cdot x_m$$

is regular. Lemma 5.12 implies that the orbits

$$G(g'' \cdot y_1, \dots, g'' \cdot y_{m-1}) \cdot (g'' \cdot y_m) \quad \text{and} \quad G(g'' \cdot y_1, \dots, g'' \cdot y_{m-1}) \cdot (g''g' \cdot x_m)$$

intersect, so there is $g''' \in G(g'' \cdot y_1, \dots, g'' \cdot y_{m-1})$ such that $g'''g''g'x_m = g''y_m$. Then the element $g = (g'')^{-1}g'''g''g'$ is as desired. □

Chapter 6

Affine cones and del Pezzo surfaces

In this chapter we study flexibility of affine cones over projective varieties, specifically over del Pezzo surfaces. We are primarily interested in their anticanonical embeddings. Note that affine cones over del Pezzo surfaces of degree ≥ 6 are toric, thereby they are flexible by [30].

We develop the cases of del Pezzo surfaces of degree 4 and 5. In case of degree 5 we prove flexibility of affine cones corresponding to polarizations defined by arbitrary very ample divisors, whereas for degree 4 we prove flexibility only for certain very ample divisors, the anticanonical one included.

As for del Pezzo surfaces of degree ≤ 3, it is proven the non-existence of any $\mathbb{G}_a$-actions on the affine cones over plurianticanonical embeddings, see [36, Theorem 1.1] for the case of degree 3 and [54, Corollary 1.8] for the case of degree ≤ 2. This answers the corresponding problem posed in [42] and [51].

In the proof we use a construction from [51], which allows to associate a regular $\mathbb{G}_a$-action on an affine cone over a projective variety Y to every open cylindrical subset of Y of some specific form. In Theorem 6.7 we provide a criterion of flexibility of an affine cone over a projective variety in terms of a transversal cover by such cylinders. We apply it to del Pezzo surfaces.

6.1 Flexibility of affine cones

It is observed in [51] that open cylindric subsets on a projective variety X give rise to one-parameter unipotent subgroups in the automorphism group of an affine cone over X. This idea is developed further in [52] and in this chapter.

Let Y be a projective variety and H be a very ample divisor on Y. A polarization of Y by H provides an embedding $Y \hookrightarrow \mathbb{P}^n$. Consider an affine cone $X = \operatorname{AffCone}_H Y \subset \mathbb{A}^{n+1}$ corresponding to this embedding. In this chapter we establish a criterion of flexibility for affine cones over affine varieties and apply it to del Pezzo surfaces of degree 4 and 5.

There is a natural homothety action of the multiplicative group $\mathbb{G}_m = \mathbb{G}_m(\mathbb{K})$ on X. It defines a grading on the algebra $\mathbb{K}[X]$.

Definition 6.1 ([51, Definitions 3.5, 3.7]). We say that an open subset U of a variety Y is a *cylinder* if $U \cong Z \times \mathbb{A}^1$, where Z is a smooth variety with $\operatorname{Pic} Z = 0$. Given a divisor $H \subset Y$, we say that a cylinder U is *H-polar* if $U = Y \setminus \operatorname{supp} D$ for some effective divisor $D \in |dH|$, where $d > 0$.

Definition 6.2. We call a subset $W \subset Y$ *invariant* with respect to a cylinder $U = Z \times \mathbb{A}^1$ if $W \cap U = \pi_1^{-1}(\pi_1(W))$, where $\pi_1 \colon U \to Z$ is the first projection of the direct product. In other words, every $\mathbb{A}^1$-fiber of the cylinder is either contained in W or does not meet W.

Definition 6.3. We say that a variety Y is *transversally covered* by cylinders U_i, $i = 1, \ldots, s$, if $Y = \bigcup U_i$ and there is no proper subset $W \subset Y$ invariant with respect to all the U_i.

The following theorem gives a criterion of flexibility for the affine cone over a projective embedding $Y \hookrightarrow \mathbb{P}^n$ corresponding to the polarization by H.

Lemma 6.4. *Let $X \subset \mathbb{A}^{n+1}$ be the affine cone over a normal projective variety $Y \subset \mathbb{P}^n$ of degree > 1. Let $G \operatorname{Aut} X$ be an algebraically generated subgroup commuting with the natural $\mathbb{G}_m$-action on X by homotheties. Assume that there is an orbit $Gx_0 \subset X$, whose image under the projection $X \setminus \{0\} \to Y$ coincides with the regular locus $Y_{\text{reg}} \subset Y$. Then $Gx = X_{\text{reg}}$.*

Proof. First of all, note that the cone X is not a linear subspace, hence the vertex of X is singular.

Since the natural $\mathbb{G}_m$-action on X by homotheties normalizes the action $G : X$, it sends G-orbits to G-orbits. So,

$$X_{\text{reg}} = \bigcup_{\lambda \in \mathbb{G}_m} \lambda G x_0 = \bigcup_{\lambda \in \mathbb{G}_m} G \lambda x_0.$$

Thus, X_{reg} is a union of G-orbits closed in X_{reg}. For any of these G-orbits its projection coincides with Y_{reg}.

Let us show that Gx_0 is open and hence coincides with X_{reg}. Assume the contrary. Then $\dim Gx_0 = \dim Y$ and the stabilizer $S \subset \mathbb{G}_m$ of the orbit Gx_0 is finite. Moreover, since the action S on Gx_0 is free, for any point $x' \in Gx_0$ the intersection $Gx_0 \cap \mathbb{G}_m x'$ is an S-orbit consisting of $|S|$ distinct points. The blow up of X at the origin is the total space of a linear bundle $\mathcal{O}_Y(-1)$ on Y. It has a natural completion — a $\mathbb{P}^1$-bundle $\hat{X} \to Y$. For a general point $x' \in Gx_0$ the intersection

$\overline{Gx_0} \cap \overline{\mathbb{G}_m x'}$, where $\overline{Z}$ denotes the closure of $Z \subset X_{\mathrm{reg}}$ in $\hat{X}$, coincides with the orbit Sx'. So, the intersection index $\overline{Gx_0} \cdot \overline{\mathbb{G}_m x'}$ equals $|S|$. Since the intersection index is constant, for any point $x' \in Gx_0$ we have $\overline{Gx_0} \cap \overline{\mathbb{G}_m x'} = Sx' \subset X_{\mathrm{reg}}$.

Let D be the union of a zero section and an infinity section of the $\mathbb{P}_1$-bundle $\hat{X} \to Y$. Then $\hat{X} \setminus D \cong X \setminus \{0\}$. Since $Sx' \subset X_{\mathrm{reg}}$ for any $x' \in Gx_0$, the intersection $D \cap \overline{Gx_0}$ is contained in the preimage of Y_{sing}. On the other hand, it is an intersection of two divisors, hence is of codimension one in D. So, it is empty.

Therefore, a quasi-affine variety $\hat{X} \setminus D$ contains a projective one $\overline{Gx_0}$. A contradiction. So, the group G acts on X_{reg} transitively. □

Proposition 6.5 ([51, Proposition 3.1.5(b)]). *If a projective variety Y possesses an H-polar cylinder U, then the affine cone $X = \mathrm{AffCone}_H Y$ admits an effective $\mathbb{G}_a$-action.*

In order to describe the latter $\mathbb{G}_a$-action we provide a straightforward proof for a slightly restricted version of this proposition.

Lemma 6.6. *Let Y be an irreducible projective variety embedded into $\mathbb{P}^n$, and U be a complement to a hyperplane section L endowed with a $\mathbb{G}_a$-action $\phi\colon \mathbb{G}_a \times U \to U$. Then the affine cone $X = \mathrm{AffCone}\, Y \subset \mathbb{A}^{n+1}$ has a homogeneous $\mathbb{G}_a$-action $\hat{\phi}\colon \mathbb{G}_a \times X \to X$ such that*

- $\phi(\mathbb{G}_a \times \{\pi(x)\}) = \pi(\hat{\phi}(\mathbb{G}_a \times \{x\}))$ *for any* $x \in \pi^{-1}(U)$.
- $\hat{\phi}$ *is the identity on* $X \setminus \pi^{-1}(U)$.

In other words, a natural projection $\pi\colon X \setminus \pi^{-1}(L) \to U$ sends any $\hat{\phi}$-orbit onto ϕ-orbit so that the preimage of any ϕ-orbit is a one-parameter family of $\hat{\phi}$-orbit.

Proof. Let Y be defined by a homogeneous ideal $I \subset \mathbb{K}[x_0, \ldots, x_n]$ which does not contain x_0, $\mathbb{K}[X] = \mathbb{K}[x_0, \ldots, x_n]/I$, and $U = \{x_0 \neq 0\} \subset Y$.

There exists a natural embedding $\rho\colon U \hookrightarrow X$, $\rho(U) = \{x_0 = 1\} \subset X$. On the other hand, $\pi\colon X \setminus \{x_0 = 0\} \to U$ is a trivial $\mathbb{C}^*$-bundle. Therefore, we may extend the $\mathbb{G}_a$-action ϕ on U to a $\mathbb{G}_a$-action $\tilde{\phi}$ on $X \setminus \{x_0 = 0\}$ generated by a homogeneous LND $\tilde{\delta}$.

There exists $d \in \mathbb{N}$ such that $x_0^d \tilde{\delta}(x_i) \in \mathbb{K}[X]$ for $i = 1, \ldots, n$. Since $x_0 \in \ker \tilde{\delta}$, a homogeneous derivation $\hat{\delta} = x_0^{d+1} \tilde{\delta}$ on X is locally nilpotent. The corresponding $\mathbb{G}_a$-action $\hat{\phi}$ is homogeneous, coincides with ϕ on $U \cong \{x_0 = 1\} \subset X$, and is identical on $\{x_0 = 0\} \subset X$. □

Theorem 6.7. *Let Y be a normal projective variety and H be a very ample divisor on Y. If there exists a transversal covering of Y_{reg} by H-polar cylinders, then the affine cone $X = \mathrm{AffCone}_H Y$ is flexible.*

Proof. The statement is obvious for $X = \mathbb{A}^{n+1}$. Thus, we may suppose that the vertex of the cone is a singular point.

By [51, Proposition 3.1.5] to an H-polar cylinder U on Y there corresponds a $\mathbb{G}_a$-action on X. Note from the explicit construction in [51, Proposition 3.1.5] that the projection $\pi\colon X \setminus \{0\} \to Y$ sends the $\mathbb{G}_a$-orbits on X to fibers of the cylinder U. The set of fixed points on $X \setminus \{0\}$ is the preimage of $Y \setminus U$. Moreover, the corresponding $\mathbb{G}_a$-actions are homogeneous.

Let $G \subset \operatorname{SAut} X$ be the subgroup spanned by these actions. Consider an orbit Gx of a smooth point $x \in X_{\text{reg}}$. The image $\pi(Gx) \subset Y_{\text{reg}}$ is invariant w.r.t. all covering cylinders. The transversality condition implies that $\pi(Gx) = Y_{\text{reg}}$. Since the group G is generated by homogeneous actions, the natural $\mathbb{G}_m$-action on X by homotheties normalizes the action G on X and sends G-orbits to G-orbits. Thus, X_{reg} is a union of G-orbits such that the projection to Y of each of these orbits coincides with Y_{reg}. Finally, by Lemma 6.4 $Gx = X_{\text{reg}}$. □

6.2 Del Pezzo surface of degree 5

Recall that all del Pezzo surfaces of degree 5 are isomorphic. Indeed, any such a surface can be obtained by blowing up the projective plane $\mathbb{P}^2$ in four points $P_1, \ldots, P_4$, no three of which are collinear [58, Theorem IV.2.5]. However, the automorphism group of the projective plane acts transitively on such 4-tuples of points.

Theorem 6.8. *Let H be an arbitrary very ample divisor on the del Pezzo surface Y of degree 5. Then the corresponding affine cone* $\operatorname{AffCone}_H Y$ *is flexible.*

The proof proceeds in several steps, see Sections 6.2.1 and 6.2.2. We let E_i denote the exceptional divisor (i.e. the (-1)-curve on Y), which is the preimage of the blown up point P_i. Let e_0 be the divisor class of a line, which contains none of the points P_i, and let e_i $(i = 1, \ldots, 4)$ be a divisor class of E_i. These classes generate the Picard group $\operatorname{Pic} Y = \langle e_0, \ldots, e_4 \rangle_{\mathbb{Z}} \cong \mathbb{Z}^5$. The intersection index defines a symmetric bilinear form on the Picard group such that the basis $\{e_0, \ldots, e_4\}$ is orthogonal, $e_0^2 = 1$, and $e_i^2 = -1$. Exceptional divisor classes are e_i and $e_0 - e_i - e_j$ for distinct $i, j \neq 0$.

By Kleiman's ampleness criterion [55, Theorem 1.4.9] the closure of the ample cone $\operatorname{Ample} Y$ is dual to the cone of effective divisors $\overline{\mathrm{NE}}(Y)$. In case of a del Pezzo surface the cone $\overline{\mathrm{NE}}(Y)$ is generated by the (-1)-curves [40, Theorem 8.2.19]. Therefore, the ample cone is defined by inequalities

$$x_0 > 0,\ x_i < 0, \qquad i = 1, \ldots, 4, \tag{6.1}$$
$$x_0 + x_i + x_j > 0, \qquad 0 \neq i \neq j \neq 0, \tag{6.2}$$

where $(x_0, \ldots, x_4) \in \operatorname{Pic} Y$. It has the following ten extremal rays

$$e_0,\ e_0 - e_j,\ 2e_0 - \sum_{i \neq 0} e_i,\ 2e_0 - \sum_{i \neq 0, j} e_i, \quad j = 1, \ldots, 4. \tag{6.3}$$

Five of them correspond to contractions $Y \to \mathbb{P}^2$ defined by 4-tuples of (-1)-curves orthogonal to the chosen ray. Any other ray defines a pencil of quadrics on Y. More precisely, the orthogonal complement to the ray contains three pairs of intersecting (-1)-curves which define the degenerate fibers of the pencil. Herewith, the class of the pencil member belongs to the chosen ray.

6.2.1 Cylinders

We have fixed above a blowing down $\varphi\colon Y \to \mathbb{P}^2$ of four pairwise disjoint (-1)-curves $E_1, \ldots, E_4$. Let $l_{ij} \subset \mathbb{P}^2$ be the line passing through the points P_i and P_j. Consider the open subset $U_1 = \varphi^{-1}(\mathbb{P}^2 \setminus (l_{12} \cup l_{34})) \subset Y$. This is a cylinder defined by a pencil of lines passing through the base point $\operatorname{Bs}(U_1) = l_{12} \cap l_{34}$. We have $U_1 \cong \mathbb{A}^1_* \times \mathbb{A}^1$, where $\mathbb{A}^1_* = \mathbb{A}^1 \setminus \{0\}$. Similarly let $U_2 = \varphi^{-1}(\mathbb{P}^2 \setminus (l_{13} \cup l_{24}))$ and $U_3 = \varphi^{-1}(\mathbb{P}^2 \setminus (l_{14} \cup l_{23}))$, see fig. 6.1. Furthermore, consider the blowings down of other 4-tuples of non-intersecting (-1)-curves on Y. There are five of them as shown on fig. 6.2. For every blowing down we define three cylinders in a similar way. Note that these cylinders are in one-to-one correspondence with the intersection points of the (-1)-curves, and the automorphism group $\operatorname{Aut} Y \cong S_5$ acts transitively on them.

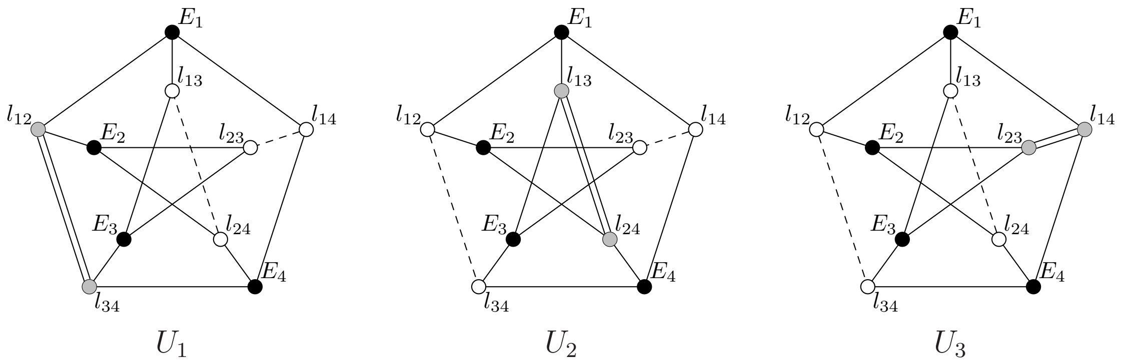

Figure 6.1
Cylinders on the incidence graph of (-1)-curves on the del Pezzo surface of degree 5. The gray and the black vertices correspond to (-1)-curves forming the complement to a cylinder. The dashed edges correspond to (-1)-curve intersections contained in the cylinder. The double edge corresponds to the base point of the cylinder.

Thus we have the cylinders $U_1, U_2, \ldots, U_{15}$ as shown on Figures 6.1 and 6.2. It is easy to check that every intersection of (-1)-curves is contained in some cylinder,

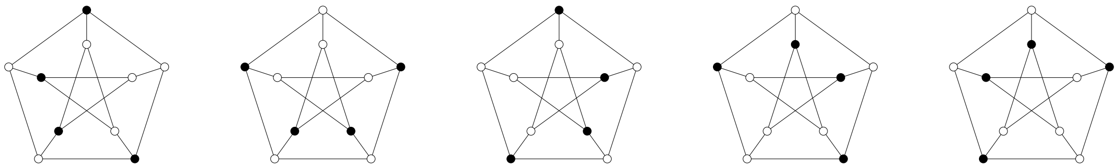

Figure 6.2
Black vertices correspond to 4-tuples of (-1)-curves. Every blowing down defines three cylinders similarly as on fig. 6.1.

hence $\bigcup U_i = Y$. We claim that there is no proper subset $W \subset Y$ invariant with respect to all 15 cylinders. Assume that there exists such a subset W. Let us fix an arbitrary point of W. It is contained in a fiber S of some cylinder, hence W contains S. Without loss of generality S is a fiber of U_1. Then a line $l = \overline{\varphi(S)} \subset \mathbb{P}^2$ passes through the base point $\mathrm{Bs}(U_1)$. Since the points $\mathrm{Bs}(U_1), \mathrm{Bs}(U_2)$, and $\mathrm{Bs}(U_3)$ do not lie on the same line, one of them does not belong to l. Suppose $\mathrm{Bs}(U_2) \notin l$. Then the fiber S intersects all but finite number of fibers of the cylinder U_2, and W contains them. So, W is dense in Y. The complement $Y \setminus W$ is also invariant with respect to all cylinders, and by the same reason it is dense in Y, a contradiction.

6.2.2 Flexible polarizations

In this subsection we show that for any ample divisor H on Y all the 15 cylinders U_i are H-polar. Consider the set of effective divisors $\{\sum_i \alpha_i E_i + \beta_1 l_{12} + \beta_3 l_{34} \mid \alpha_i, \beta_i > 0\}$ whose support is the complement to U_1. The image of this set in the Picard group is an open cone C, whose extremal rays are $e_1,\ e_2,\ e_3,\ e_4,\ e_0 - e_1 - e_2$, and $e_0 - e_3 - e_4$. It is easy to check that the primitive vectors of the ample cone (6.3) can be expressed as linear combinations with non-negative rational coefficients of the primitive vectors of the cone C. Therefore the cylinder U_1 is H-polar for any ample divisor H. By means of automorphisms from $\mathrm{Aut}\, Y$ we may translate U_1 to any other cylinder U_i. Hence the cylinders U_i are H-polar for any ample divisor H. Using Theorem 6.7 we obtain the assertion. Now Theorem 6.8 is proved.

6.3 Del Pezzo surfaces of degree 4

Every del Pezzo surface of degree 4 can be obtained by blowing up a projective plane $\mathbb{P}^2$ in five points, where no three are collinear. The moduli space of such surfaces is two-dimensional.

By E_i we denote the (-1)-curve which is the preimage of the blown up point P_i. As before, let e_0 be the divisor class of a line which does not contain the blown up points, and e_i $(i = 1, \ldots, 5)$ be the divisor class of E_i. A set $\{e_0, \ldots, e_5\}$ forms an orthogonal basis of the Picard group $\operatorname{Pic} Y \cong \mathbb{Z}^6$, and $e_0^2 = 1$, $e_i^2 = -1$. The classes of (-1)-curves are e_i, $e_0 - e_i - e_j$, $2e_0 - \sum_{k \neq 0} e_k$ for any pair of distinct indices $i, j \neq 0$. The ample cone is defined by inequalities

$$\begin{aligned} x_0 > 0,\ x_i &< 0 \qquad i = 1, \ldots, 5, && (6.4)\\ x_0 + x_i + x_j &> 0, \qquad 0 \neq i \neq j \neq 0, && (6.5)\\ 2x_0 + x_1 + \ldots + x_5 &> 0, && (6.6) \end{aligned}$$

where $(x_0, \ldots, x_5) \in \operatorname{Pic} Y$. Its extremal rays are

$$e_0,\ e_0 - e_j,\ 2e_0 - \sum_{k \neq 0, i} e_k,\ 2e_0 - \sum_{k \neq 0, i, j} e_k,\ \text{and } 3e_0 - \sum_{k \neq 0} e_k - e_i \tag{6.7}$$

for any pair of distinct indices $i, j \in \{1, \ldots, 5\}$. Similarly to the case of a del Pezzo surface of degree 5, sixteen of the extremal rays correspond to different blowdowns $Y \to \mathbb{P}^2$, and the ten remaining rays correspond to different pencils of quadrics on Y.

6.3.1 Cylinders

Let us fix some (-1)-curve C_1 and consider the blowing down $\sigma_1 \colon Y \to \mathbb{P}^2$ of five (-1)-curves $F_1, \ldots, F_5$ that meet C_1, see fig. 6.3. This blowing down is well defined since the contracted divisors do not intersect. The image $\sigma_1(C_1)$ is a smooth quadric c passing through the blown down points $Q_1, \ldots, Q_5$. Take an arbitrary line $l \subset \mathbb{P}^2$ which is tangent to c at a point different from $Q_1, \ldots, Q_5$. A quadric pencil in $\mathbb{P}^2$ generated by divisors c and $2l$ determines a cylinder $U \cong \mathbb{A}^1_* \times \mathbb{A}^1 \subset Y$ whose complement is the complete preimage of the support of the divisor $c + 2l$ on $\mathbb{P}^2$. Denote by $\mathcal{U}_{C_1}$ the family of all such cylinders in Y for all such tangents l. Note that $Y \setminus \bigcup_{U \in \mathcal{U}_{C_1}} U$ is a union of C_1 and the exceptional divisors F_i $(i = 1, \ldots, 5)$. Apply this construction to the (-1)-curves $C_2, \ldots, C_5$, which form a 5-cycle along with C_1 on the incidence graph as shown on fig. 6.3. Overall we obtain five cylinder families $\mathcal{U}_{C_1}, \ldots, \mathcal{U}_{C_5}$. It is easily seen that their union covers Y.

Let W be a proper subset of Y invariant with respect to the cylinders of all families, and let $w \in W$ be an arbitrary point. We may suppose that w belongs to a cylinder of the family $\mathcal{U}_{C_1}$. Then the image $\sigma_1(W) \subset \mathbb{P}^2$ is invariant with respect to the cylinder family $\{\sigma_1(U) \mid U \in \mathcal{U}_{C_1}\}$. Note that every cylinder of this family is a complement to the quadric c and its tangent line. It is well known that given a quadric and two points outside it we can find a quadric passing through these two points and tangent to the given quadric. Therefore, for almost every point $x \in \mathbb{P}^2 \setminus c$ there exists a fiber of some cylinder which contains x and $\sigma_1(w)$. Namely, x must not lie on the tangent line to c passing through $\sigma_1(w)$ as well as on the quadrics which are tangent to c at blown down points and contain $\sigma_1(w)$. Thus W is dense in Y. Similarly, $Y \setminus W$ is dense in Y, a contradiction. Finally, the families $\mathcal{U}_{C_1}, \dots, \mathcal{U}_{C_5}$ form a transversal cover of Y.

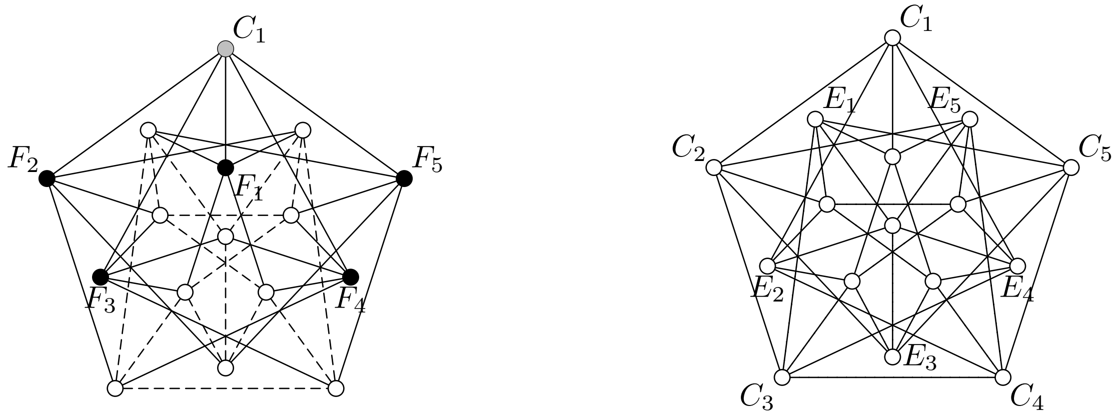

Figure 6.3
The incidence graph of (-1)-curves on a del Pezzo surface of degree 4. On the left the gray vertex corresponds to the quadric preimage C_1 and black vertices correspond to the contracted (-1)-curves. The dashed edges correspond to (-1)-curve intersections contained in the cylinders of a family. Four other families corresponding to $C_2, \dots, C_5$ are obtained symmetrically by the graph rotations.

6.3.2 Flexible polarizations

Ample divisors H such that cylinders of the family $\mathcal{U}_{C_i}$ are H-polar, are exactly the ample divisors in the set $\operatorname{Ample} Y \cap \{\alpha_1 F_1 + \ldots + \alpha_5 F_5 + \alpha_6 C_i + \alpha_7 \sigma_i^{-1}(l) \mid \alpha_j > 0\}$ in $\operatorname{Pic} Y$. This set is an open cone which we denote by $\operatorname{Ample}(C_i, Y)$. It does not depend on a choice of a tangent line l since it does not contain blown up points by definition. Then the set of the divisors H such that cylinders in $\bigcup_i \mathcal{U}_{C_i}$ are H-polar is an open cone $\bigcap_i \operatorname{Ample}(C_i, Y)$. A computation shows that it has exactly 72 extremal rays, which can be expressed as

$$\begin{aligned}
&e_0, && 9e_0 - 5e_{i_1} - e_{i_2} - 2e_{i_3} - 4e_{i_4} - 3e_{i_5},\\
&4e_0 - 2e_{i_1} - 2e_{i_2} - e_{i_3} - e_{i_4} - e_{i_5}, && 9e_0 - 4e_{i_1} - 4e_{i_2} - 4e_{i_3} - 2e_{i_4} - 2e_{i_5},\\
&5e_0 - 2e_{i_1} - 2e_{i_2} - e_{i_3} - 3e_{i_4} - e_{i_5}, && 11e_0 - 6e_{i_1} - 2e_{i_2} - 2e_{i_3} - 4e_{i_4} - 4e_{i_5},\\
&5e_0 - 2e_{i_1} - 2e_{i_2} - 2e_{i_3} - 2e_{i_4}, && 11e_0 - 6e_{i_1} - 4e_{i_2} - 4e_{i_3} - 2e_{i_4} - 2e_{i_5},\\
&5e_0 - 2e_{i_1} - 2e_{i_2} - 2e_{i_3} - 2e_{i_4} - 2e_{i_5}, && 11e_0 - 6e_{i_1} - 2e_{i_2} - 4e_{i_3} - 4e_{i_4} - 4e_{i_5},\\
&6e_0 - 2e_{i_1} - 2e_{i_2} - 3e_{i_3} - e_{i_4} - 3e_{i_5}, && 11e_0 - 6e_{i_1} - 4e_{i_2} - 4e_{i_3} - 4e_{i_4} - 2e_{i_5},\\
&7e_0 - 4e_{i_1} - 2e_{i_2} - 2e_{i_3} - 2e_{i_4} - 2e_{i_5}, && 15e_0 - 8e_{i_1} - 2e_{i_2} - 4e_{i_3} - 6e_{i_4} - 6e_{i_5},\\
&9e_0 - 5e_{i_1} - 3e_{i_2} - 4e_{i_3} - 2e_{i_4} - 1e_{i_5}, && 15e_0 - 8e_{i_1} - 6e_{i_2} - 6e_{i_3} - 4e_{i_4} - 2e_{i_5},
\end{aligned}$$

where the tuple $(i_1, \ldots, i_5)$ runs over all cyclic permutations of $(1, 2, 3, 4, 5)$.

It is easily seen that the anticanonical divisor $(-K_Y)$ is contained in $\bigcap_i \mathrm{Ample}(C_i, Y)$. Similarly to Theorem 6.8 we obtain the following result.

Theorem 6.9. *Let Y be a del Pezzo surface of degree 4, and H be a very ample divisor in the open cone $\bigcap_{i=1}^5 \mathrm{Ample}(C_i, Y)$. Then the affine cone $\mathrm{AffCone}_H Y$ is flexible. In particular, this holds for the anticanonical divisor $H = -K_Y$.*

We have identified a subcone of the ample cone such that the very ample divisors contained in this subcone define a flexible affine cone. However, this subcone is strictly contained in the ample cone. For example, the ample divisor class $8e_0 - 2e_1 - 4e_2 - e_3 - e_4 - 3e_5$ lies outside of that subcone. Thus the flexibility problem remains open for the affine cones over the del Pezzo surfaces of degree 4 polarized by an arbitrary very ample divisor.

Chapter 7

Torsors and A-covered varieties

In this chapter we present a joint work with Ivan Arzhantsev and Hendrik Suess [31].

7.1 Introduction

Universal torsors were introduced by Colliot-Thélène and Sansuc in the framework of arithmetic geometry to investigate rational points on algebraic varieties, see [37], [38], [65]. In the last years they were used to obtain positive results on Manin's Conjecture. Another source of interest is Cox's paper [39], where an explicit description of the universal torsor over a toric variety is given. This approach had an essential impact in toric geometry. For generalizations and relations to Cox rings, see [47], [33], [34], [46], [28].

Let X be a smooth algebraic variety. Assume that the divisor class group $\mathrm{Cl}(X)$ is a lattice of rank r. The universal torsor $q : \widehat{X} \to X$ is a locally trivial H-principal bundle with certain characteristic properties, where H is an algebraic torus of dimension r, see [65, Section 1]; here $\widehat{X}$ is a smooth quasiaffine algebraic variety.

The aim of this chapter is to show that under certain restrictions on X the automorphism group $\mathrm{Aut}(\widehat{X})$ acts on $\widehat{X}$ infinitely transitively. We use a construction of [51] to show that open cylindric subsets on X define one-parameter unipotent subgroups L_i in $\mathrm{Aut}(\widehat{X})$. It turns out that the subgroup generated by L_i acts on $\widehat{X}$ transitively.

This chapter is organized as follows. In Section 7.2 we recall basic definitions and facts on Cox rings and universal torsors. In Section 7.3 we recall some definitions from Section 6.1 and show that if X is a smooth algebraic variety with a free finitely generated divisor class group $\mathrm{Cl}(X)$, which is transversally covered by cylinders, then the group $\mathrm{SAut}(\widehat{X})$ acts on the universal torsor $\widehat{X}$ transitively.

As a particular case, in Section 7.4 we study A-covered varieties, i.e. varieties covered by open subsets isomorphic to the affine space of the same dimension. Clearly, any A-covered variety is smooth and rational. We list wide classes of A-covered varieties including smooth complete toric or, more generally, spherical varieties, smooth rational projective surfaces, and some Fano threefolds. It is shown that the condition to be A-covered is preserved under passing to vector bundles and their projectivizations as well as to the blow up in a linear subvariety.

In Section 7.5 we summarize our results on universal torsors and infinite transitivity. Theorem 7.14 claims that if X is an A-covered algebraic variety of dimension at least 2, then $\mathrm{SAut}(\widehat{X})$ acts on the universal torsor $\widehat{X}$ infinitely transitively. If the Cox ring $\mathcal{R}(X)$ is finitely generated, then the total coordinate space $\overline{X} := \mathrm{Spec}\,\mathcal{R}(X)$ is a factorial affine variety, the group $\mathrm{SAut}(\overline{X})$ acts on $\overline{X}$ with an open orbit O, and the action of $\mathrm{SAut}(\overline{X})$ on O is infinitely transitive, see Theorem 5.7. In particular, the Makar-Limanov invariant of $\overline{X}$ is trivial, see Corollary 7.16.

We work over an algebraically closed field $\mathbb{K}$ of characteristic zero.

7.2 Preliminaries on Cox rings and universal torsors

Let X be a normal algebraic variety with free finitely generated divisor class group $\mathrm{Cl}(X)$. Denote by $\mathrm{WDiv}(X)$ the group of Weil divisors on X and fix a subgroup $K \subset \mathrm{WDiv}(X)$ such that the canonical map $c\colon K \to \mathrm{Cl}(X)$ sending $D \in K$ to its class $[D] \in \mathrm{Cl}(X)$ is an isomorphism. We define the *Cox sheaf* associated to K to be

$$\mathcal{R} := \bigoplus_{[D]\in\mathrm{Cl}(X)} \mathcal{R}_{[D]}, \qquad \mathcal{R}_{[D]} := \mathcal{O}_X(D),$$

where $D \in K$ represents $[D] \in \mathrm{Cl}(X)$ and the multiplication in $\mathcal{R}$ is defined by multiplying homogeneous sections in the field of rational functions $\mathbb{K}(X)$. The sheaf $\mathcal{R}$ is a quasicoherent sheaf of normal integral K-graded $\mathcal{O}_X$-algebras and, up to isomorphy, it does not depend on the choice of the subgroup $K \subset \mathrm{WDiv}(X)$, see [28, Construction I.4.1.1]. The *Cox ring* of X is the algebra of global sections

$$\mathcal{R}(X) := \bigoplus_{[D]\in\mathrm{Cl}(X)} \mathcal{R}_{[D]}(X), \qquad \mathcal{R}_{[D]}(X) := \Gamma(X, \mathcal{O}_X(D)).$$

Let us assume that X is a smooth variety with only constant invertible functions. Then the sheaf $\mathcal{R}$ is locally of finite type, and the relative spectrum $\mathrm{Spec}_X\,\mathcal{R}$ is a quasiaffine variety $\widehat{X}$, see [28, Corollary I.3.4.6]. We have $\Gamma(\widehat{X}, \mathcal{O}) \cong \mathcal{R}(X)$, and the ring $\mathcal{R}(X)$ is a unique factorization domain with only constant invertible elements, see [28, Proposition I.4.1.5]. Since the sheaf $\mathcal{R}$ is K-graded, the variety $\widehat{X}$ carries a natural action of the torus $H := \operatorname{Spec} \mathbb{K}[K]$. The projection $q\colon \widehat{X} \to X$ is called the *universal torsor* over the variety X. By [28, Remark I.3.2.7], the morphism $q\colon \widehat{X} \to X$ is a locally trivial H-principal bundle. In particular, the torus H acts on $\widehat{X}$ freely.

Lemma 7.1. *Let X be a normal variety. Assume that there is an open subset U on X which is isomorphic to the affine space $\mathbb{A}^n$. Then any invertible function on X is constant and the group $\mathrm{Cl}(X)$ is freely generated by classes $[D_1], \ldots, [D_k]$ of the prime divisors such that*

$$X \setminus U = D_1 \cup \ldots \cup D_k.$$

Proof. The restriction of an invertible function to U is constant, so the function is constant. Since U is factorial, any Weil divisor on X is linearly equivalent to a divisor whose support does not intersect U. This shows that the group $\mathrm{Cl}(X)$ is generated by $[D_1], \ldots, [D_k]$.

Assume that $a_1D_1 + \ldots + a_kD_k = \operatorname{div}(f)$ for some $f \in \mathbb{K}(X)$. Then f is a regular invertible function on U and thus f is a constant. This shows that the classes $[D_1], \ldots, [D_k]$ generate the group $\mathrm{Cl}(X)$ freely. □

The Cox ring $\mathcal{R}(X)$ and the relative spectrum $q\colon \widehat{X} \to X$ can be defined and studied under weaker assumptions on the variety X, see [28, Chapter I]. But in this chapter we are interested in smooth varieties with free finitely generated divisor class group.

Assume that the Cox ring $\mathcal{R}(X)$ is finitely generated. Then we may consider the *total coordinate space* $\overline{X} := \operatorname{Spec} \mathcal{R}(X)$. This is a factorial affine H-variety. By [28, Construction I.6.3.1], there is a natural open H-equivariant embedding $\widehat{X} \hookrightarrow \overline{X}$ such that the complement $\overline{X} \setminus \widehat{X}$ is of codimension at least two.

7.3 Cylinders and $\mathbb{G}_a$-actions

The following definition is taken from [51], see also [52].

Definition 7.2. Let X be an algebraic variety and U be an open subset of X. We say that U is a *cylinder* if $U \cong Z \times \mathbb{A}^1$, where Z is an irreducible affine variety with $\mathrm{Cl}(Z) = 0$.

Proposition 7.3. *Let X be a smooth algebraic variety with a free finitely generated divisor class group $\mathrm{Cl}(X)$, $q\colon \widehat{X} \to X$ be the universal torsor, and $U \cong Z \times \mathbb{A}^1$ be a cylinder in X. Then there is an action $\mathbb{G}_a \times \widehat{X} \to \widehat{X}$ such that* *(i) the set of $\mathbb{G}_a$-fixed points is $\widehat{X} \setminus q^{-1}(U)$;*

(ii) for any point $y \in q^{-1}(U)$ we have $q(L \cdot y) = \{z\} \times \mathbb{A}^1$ for some $z \in Z$, where L is the image of $\mathbb{G}_a$ in $\operatorname{Aut}(\widehat{X})$.

Proof. Since $\operatorname{Cl}(U) \cong \operatorname{Cl}(Z) = 0$, we have an isomorphism $q^{-1}(U) \cong Z \times \mathbb{A}^1 \times H$ compatible with the projection q, see [28, Remark I.3.2.7]. Thus the subset $q^{-1}(U)$ admits a $\mathbb{G}_a$-action

$$a \cdot (z, t, h) = (z, t + a, h), \quad z \in Z,\ t \in \mathbb{A}^1,\ h \in H,$$

with property (ii). Denote by D the locally nilpotent derivation on $\Gamma(U, \mathcal{O})$ corresponding to this action.

Our aim is to extend the action to $\widehat{X}$. Since the open subset $q^{-1}(U)$ is affine, its complement $\widehat{X} \setminus q^{-1}(U)$ is a divisor Δ in $\widehat{X}$. We can find a function $f \in \Gamma(\widehat{X}, \mathcal{O})$ such that $\Delta = \operatorname{div}(f)$. In particular,

$$\Gamma(q^{-1}(U), \mathcal{O}) = \Gamma(\widehat{X}, \mathcal{O})[1/f].$$

Since f has no zero on any $\mathbb{G}_a$-orbit on $q^{-1}(U)$, it is constant along orbits, and f lies in $\operatorname{Ker} D$.

Lemma 7.4. *Let Y be an irreducible quasiaffine variety,*

$$Y = \bigcup_{i=1}^{s} Y_{g_i}, \qquad g_i \in \Gamma(Y, \mathcal{O}),$$

be an open covering by principle affine subsets, and let

$$\Gamma(Y_{g_i}, \mathcal{O}) = \mathbb{K}[c_{i1}, \ldots, c_{ir_i}][1/g_i]$$

for some $c_{ij} \in \Gamma(Y, \mathcal{O})$. Consider a finitely generated subalgebra C in $\Gamma(Y, \mathcal{O})$ containing all the functions g_i and c_{ij}. Then the natural morphism $Y \to \operatorname{Spec} C$ is an open embedding.

Proof. Notice that $\Gamma(Y_{g_i}, \mathcal{O}) = \Gamma(Y, \mathcal{O})[1/g_i] = C[1/g_i]$. This shows that the morphism $Y \to \operatorname{Spec} C$ induces isomorphisms $Y_{g_i} \cong (\operatorname{Spec} C)_{g_i}$. □

Let $Y = \widehat{X}$ and $\widehat{X} \hookrightarrow \operatorname{Spec} C$ be an affine embedding as in Lemma 7.4 with $f \in C$. A finite generating set of the algebra C is contained in a finite dimensional D-invariant subspace W of $\Gamma(q^{-1}(U), \mathcal{O})$. Replacing D with $f^m D$ we may assume that W is contained in $\Gamma(\widehat{X}, \mathcal{O})$. We enlarge C and assume that it is generated by W. Then C is an $(f^m D)$-invariant finitely generated subalgebra in $\Gamma(\widehat{X}, \mathcal{O})$ and we have an open embedding $\widehat{X} \hookrightarrow \operatorname{Spec} C =: \widetilde{X}$.

Replacing $f^m D$ with $D' := f^{m+1}D$, we obtain a locally nilpotent derivation D' on C such that $D'(C)$ is contained in fC. The corresponding $\mathbb{G}_a$-action on $\widetilde{X}$ fixes all points on $\operatorname{div}(f)$ and has the same orbits on $q^{-1}(U)$. Hence the subset $\widehat{X} \subset \widetilde{X}$ is $\mathbb{G}_a$-invariant and the restriction of the action to $\widetilde{X}$ has the desired properties. The proof of Proposition 7.3 is completed. □

Remark 7.5. Under the assumption that the algebra $\Gamma(\widetilde{X}, \mathcal{O})$ is finitely generated the proof of Proposition 7.3 is much simpler.

The following definitions appeared in [61].

Definition 7.6. Let X be a variety and $U \cong Z \times \mathbb{A}^1$ be a cylinder in X. A subset W of X is said to be *U-invariant* if $W \cap U = p_1^{-1}(p_1(W \cap U))$, where $p_1 \colon U \to Z$ is the projection to the first factor. In other words, every $\mathbb{A}^1$-fiber of the cylinder is either contained in W or does not meet W.

Definition 7.7. We say that a variety X is *transversally covered* by cylinders U_i, $i = 1, \ldots, s$, if $X = \bigcup_{i=1}^s U_i$ and there is no proper subset $W \subset X$ invariant under all U_i.

Proposition 7.8. *Let X be a smooth algebraic variety with a free finitely generated divisor class group* $\operatorname{Cl}(X)$ *and $q \colon \widehat{X} \to X$ be the universal torsor. Assume that X is transversally covered by cylinders. Then the group* $\operatorname{SAut}(\widehat{X})$ *acts on $\widehat{X}$ transitively.*

Proof. Consider a $\mathbb{G}_a$-action on $\widehat{X}$ associated with the cylinder U_i as in Proposition 7.3. Let L_i be the corresponding $\mathbb{G}_a$-subgroup in $\operatorname{SAut}(\widehat{X})$ and G be the subgroup of $\operatorname{SAut}(\widehat{X})$ generated by all the L_i. By construction, the subgroups L_i and thus the group G commute with the torus H.

Let S be a G-orbit on $\widehat{X}$. By Proposition 7.3, the projection $q(S)$ is invariant under all the cylinders U_i, and thus $q(S)$ coincides with X. Let H_S be the stabilizer of the subset S in H. Then the map $H \times S \to \widehat{X}$, $(h, x) \mapsto hx$, is surjective and its image is isomorphic to $(H/H_S) \times S$. Since H/H_S is a torus and the variety $\widehat{X}$ has only constant invertible functions, we conclude that $H_S = H$ and thus $S = \widehat{X}$. This shows that G, and hence $\operatorname{SAut}(\widehat{X})$, acts on $\widehat{X}$ transitively. □

Remark 7.9. By Theorem 5.7 the action of $\operatorname{SAut}(\widehat{X})$ on $\widehat{X}$ in Proposition 7.8 is infinitely transitive.

7.4 A-covered varieties

The affine space $\mathbb{A}^n$ admits n coordinate cylinder structures $\mathbb{A}^{n-1} \times \mathbb{A}^1$, and the covering of $\mathbb{A}^n$ by these cylinders is transversal. This elementary observation motivates the following definition.

Definition 7.10. An irreducible algebraic variety X is said to be *A-covered* if there is an open covering $X = U_1 \cup \ldots \cup U_r$, where every chart U_i is isomorphic to the affine space $\mathbb{A}^n$.

A choice of such a covering together with isomorphisms $U_i \cong \mathbb{A}^n$ is called an *A-atlas* of X. A subvariety Z of an A-covered variety X is called *linear* with respect to an A-atlas, if it is linear in all charts, i.e. $Z \cap U_i$ is a linear subspace in $U_i \cong \mathbb{A}^n$. Any A-covered variety is rational, smooth, and by Lemma 7.1 the group $\mathrm{Pic}(X) = \mathrm{Cl}(X)$ is finitely generated and free.

Clearly, the projective space $\mathbb{P}^n$ is A-covered. This fact can be generalized in several ways.

(1) Every smooth complete toric variety X is A-covered.

(2) Every smooth rational complete variety with a torus action of complexity one is A-covered; see [31, Theorem 5].

(3) Let G be a semisimple algebraic group and let P be a parabolic subgroup of G. Then the flag variety G/P is A-covered. Indeed, a maximal unipotent subgroup N of G acts on G/P with an open orbit U isomorphic to an affine space. Since G acts on G/P transitively, we obtain the desired covering.

    Remark 7.11. Consider an A-atlas of G/P generated from U by G-actions. Then any irreducible subvariety Z linear in U is linear w.r.t. to the A-atlas.

(4) More generally, every complete smooth spherical variety is A-covered, see [35, Corollary 1.5].

(5) The Fano threefolds $\mathbb{P}^3$, Q, V_5 and an element of the family V_{22} are known to be A-covered. Moreover, there are no other types of A-covered Fano threefolds of Picard number 1 by [44]. In particular, the Fano threefolds V_{12}, V_{16}, V_{18} and V_4 from Iskovskikh's classification [50] are rational but not A-covered.

(6) The product of two A-covered varieties is again A-covered.

(7) More generally, every vector bundle over $\mathbb{A}^n$ trivializes, and so the total spaces of vector bundles over A-covered varieties are A-covered. The same holds for their projectivizations.

(8) If a variety X is A-covered and X' is a blow up of X at some point $p \in X$, then X' is A-covered.

(9) In particular, all smooth projective rational surfaces are obtained either from $\mathbb{P}^2$, $\mathbb{P}^1 \times \mathbb{P}^1$ or from the Hirzebruch surfaces F_n by a sequence of blow ups of points, and so they are A-covered. [1]

(10) We may generalize the blow up example as follows. The blow up of X in a linear subvariety Z is A-covered. Moreover, the strict transforms of linear subvarieties, which either contain Z or do not intersect with it, are linear again (with the choice of an appropriate A-atlas). Hence, we may iterate this procedure.

Proof of (10). We consider a chart U of the A-covering of X. We may assume that we blow up $\mathbb{A}^n = U$ in the linear subspace given by $x_1 = \ldots = x_k = 0$. By definition, the blow up X' is given in the product $\mathbb{A}^n \times \mathbb{P}^{k-1}$ by equations $x_i z_j = x_j z_i$, where $1 \leq i, j \leq k$. If the homogeneous coordinate z_j equals 1 for some $j = 1, \ldots, k$, then $x_i = x_j z_i$, and we are in the open chart V_j with independent coordinates x_j, x_s with $s > k$, and z_i, $i \neq j$. So the variety X' is covered by k such charts.

Let L be a linear subspace in U containing $[x_1 = \ldots = x_k = 0]$ and given by linear equations $f_i(x_1, \ldots, x_k) = 0$. The strict transform of L is given in V_j by the equations $f_i(z_1, \ldots, z_{j-1}, 1, z_{j+1}, \ldots, z_k) = 0$. After a change of variables $x_j \mapsto x_j - 1$ these equations become linear.

Finally, if a linear subvariety Z' does not meet the linear subvariety Z, then Z' does not intersect charts of our atlas that intersect Z, and the assertion follows. ☐

Example 7.12. Consider a smooth quadric threefold Q. Choose two points and a conic passing through them. Then these are linear subvarieties of Q with respect to an appropriate atlas. Hence, the iterated blow up in the points, first, and then in the strict transform of the conic is A-covered.

We may use the above observations to take a closer look at Fano threefolds.

Proposition 7.13. *In the classification of Iskovskikh [50] and Mori-Mukai [59] we have the following (possibly non-complete) list of A-covered Fano threefolds:*

(a) $\mathbb{P}^3$, Q, V_5, *(at least) one element* V'_{22} *of the family* V_{22}*;*

(b) 2.33-2.36, 3.26-3.31, 4.9-4.11, 5.2, 5.3;

(c) 2.29, 2.30, 2.31, 2.32, 3.8, 3.18-3.23, 3.24, 4.4, 4.7, 4.8, (at least) one element of the families 2.24, 3.8 and 3.10 respectively;

[1] This property of smooth rational surfaces was used in [51, Proposition 3.3] in order to prove the existence of an affine cone with a $\mathbb{G}_a$-action.

(d) 5.3-5.8;

(e) (at least) one element of the family 2.26.

Proof. The list in (a) is the same as that in (5). The list in (b) consists exactly of the toric Fano threefolds. The varieties in (c) admit a 2-torus action. This can be seen more or less directly from the description given in [59]. For some of them we get alternative proofs of the A-coveredness by (3), (7) and (10). The varieties in (d) are products of del Pezzo surfaces (which are rational) and $\mathbb{P}^1$. The variety in (e) is obtained from V_5 by blow up in a linear subvariety, as explained in (10). $\square$

7.5 Main results

The following theorem summarizes our results on universal torsors and infinite transitivity.

Theorem 7.14. *Let X be an A-covered algebraic variety of dimension at least 2 and $q\colon \widehat{X} \to X$ be the universal torsor. Then the group $\mathrm{SAut}(\widehat{X})$ acts on the quasiaffine variety $\widehat{X}$ infinitely transitively.*

Proof. If X is covered by m open charts isomorphic to $\mathbb{A}^n$, and every chart is equipped with n transversal cylinder structures, then the covering of X by these mn cylinders is transversal. By Proposition 7.8, the group $\mathrm{SAut}(\widehat{X})$ acts on $\widehat{X}$ transitively. Theorem 5.7 yields that the action is infinitely transitive. $\square$

Theorem 7.14 provides many examples of quasiaffine varieties with rich symmetries. In particular, if X is a del Pezzo surface, a description of the universal torsor $q\colon \widehat{X} \to X$ may be found in [32], [63], [64]. It follows from Theorem 7.14 that the group $\mathrm{SAut}(\widehat{X})$ acts on $\widehat{X}$ infinitely transitively.

Let X be the blow up of nine points in general position on $\mathbb{P}^2$. By [60], the Cox ring $\mathcal{R}(X)$ is not finitely generated, and thus $\widehat{X}$ is a quasiaffine variety with a non-finitely generated algebra of regular functions $\Gamma(\widehat{X}, \mathcal{O})$. Theorem 7.14 works in this case as well.

Theorem 7.15. *Let X be an A-covered algebraic variety of dimension at least 2. Assume that the Cox ring $\mathcal{R}(X)$ is finitely generated. Then the total coordinate space $\overline{X} := \mathrm{Spec}\ \mathcal{R}(X)$ is a factorial affine variety, the group $\mathrm{SAut}(\overline{X})$ acts on $\overline{X}$ with an open orbit O, and the action of $\mathrm{SAut}(\overline{X})$ on O is infinitely transitive.*

Proof. Lemma 7.1 shows that the group $\mathrm{Cl}(X)$ is finitely generated and free, hence the ring $\mathcal{R}(X)$ is a unique factorization domain, see [28, Proposition I.4.1.5]. Since

$$\Gamma(\overline{X}, \mathcal{O}) = \mathcal{R}(X) \cong \Gamma(\widehat{X}, \mathcal{O}),$$

any $\mathbb{G}_a$-action on $\widehat{X}$ extends to $\overline{X}$. We conclude that $\widehat{X}$ is contained in one $\mathrm{SAut}(\overline{X})$-orbit O on $\overline{X}$, the action of $\mathrm{SAut}(\overline{X})$ on O is infinitely transitive, and by [29, Proposition 1.3] the orbit O is open in $\overline{X}$. $\square$

Recall from [43] that the *Makar-Limanov invariant* $\mathrm{ML}(Y)$ of an affine variety Y is the intersection of the kernels of all locally nilpotent derivations on $\Gamma(Y, \mathcal{O})$. In other words $\mathrm{ML}(Y)$ is the subalgebra of all $\mathrm{SAut}(Y)$-invariants in $\Gamma(Y, \mathcal{O})$. Similarly to as in [56] the *field Makar-Limanov invariant* $\mathrm{FML}(Y)$ is the subfield of $\mathbb{K}(Y)$ which consists of all rational $\mathrm{SAut}(Y)$-invariants. If the field Makar-Limanov invariant is trivial, that is, if $\mathrm{FML}(Y) = \mathbb{K}$, then so is $\mathrm{ML}(Y)$, but the converse is not true in general.

Corollary 7.16. *Under the assumptions of Theorem 7.15 the field Makar-Limanov invariant $FML(\overline{X})$ is trivial.*

Proof. By Theorem 7.15, the group $\mathrm{SAut}(\overline{X})$ acts on $\overline{X}$ with an open orbit. The claim follows from Proposition 5.5. $\square$

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